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III. Illustrative of Mr. C. V. Boys's Paper on the Drawing of Curves by their Curvature.

PROCEEDINGS

OF

THE PHYSICAL SOCIETY

OF LONDON.

MARCH 1893.

I. *Notes on the Construction of a Colour Map.*

By WALTER BAILY, *M.A.*[*]

1. IN my paper on this subject, read before the Physical Society on April 8 last[†], in discussing the method of plotting out the curve giving the mixtures of two spectrum colours, I stated that points in the curve lying in the abnormal or imaginary regions could not be determined by experiment. This I now find is a mistake.

Let S_1 and S_2 be spectrum colours, of which the mixtures are to be represented by points on the curve. The part of the curve lying in the imaginary and abnormal regions is obtained by *subtracting* light of one colour from light of the other colour, and equating the resulting colour to the mixture (if I may use the expression) obtained by *subtracting* white from some other spectrum colour S.

We have

$$n_1S_1 - n_2S_2 = S - xW.$$

This equation does not represent any possible physical experiment, but it may be transformed, without altering the value of any of the quantities, into

$$S + n_2S_2 = n_1S_1 + xW.$$

This last equation means:—Make a colour-patch of S and

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[*] Read October 28, 1892.

[†] Proc. Phys. Soc. vol. xi. p. 323.

S_2, and a colour-patch of S_1 and W, and vary the quantities of S_1, S_2, and W until the two patches are similar in colour, and then measure the quantity of white. A line drawn to the left from the position of S, and proportional in length to the quantity of white, will define the position of the required point in the curve.

2. In the map the region on the right of the line of spectrum colours is occupied by all the colours obtained by mixing spectrum colours and white. I suggest that this should be called the Spectral Region. The equation for determining points on the mixture curves in the Spectral Region is

$$n_1S_1+n_2S_2=S+xW.$$

The region to the left of the abnormal and imaginary regions is occupied by colours complementary to those in the Spectral Region. I suggest that this should be called the Complementary Region. The equation for determining points on the mixture curves in the Complementary Region is

$$S+n_1S_1+n_2S_2=xW.$$

The equation for determining points on the mixture curves in the Imaginary and Abnormal Regions has just been shown to be (for one part of the curve)

$$S+n_2S_2=n_1S_1+xW,$$

and it is obvious that for the other part it will be

$$S+n_1S_1=n_2S_2+xW.$$

3. In defining a colour as compounded of a spectrum colour and white, we have to determine both quality and quantity of the white light.

A colour may be defined by three spectrum colours by means of the equation

$$n_1S_1+n_2S_2=n_3S_3+xW,$$

or of the equation

$$n_1S_1+n_2S_2+n_3S_3=xW.$$

The colour is that obtained when a patch formed in accordance with one side of the equation is made identical with a patch formed in accordance with the other. In this case quantity of white need not be defined, but quality must be defined.

If a colour be defined by four spectrum colours the definition is independent of any convention as to the quality of white light. Let S_1, S_2, S_3, S_4 be four spectrum colours whose wave-lengths are in order of magnitude, then the equation

$$n_1S_1 + n_3S_3 = n_2S_2 + n_4S_4$$

defines a certain colour, which can be obtained by making a patch of S_1 and S_3 identical in colour with one of S_2 and S_4.

II. *Experiments in Electric and Magnetic Fields, Constant and Varying. By* Messrs. RIMINGTON *and* WYTHE SMITH.*

[Plate I.]

THE object the authors have in bringing these phenomena before the Society is not to establish any new theory, nor to show imperfections in present theories, but to exhibit experiments which help the actions taking place in a dielectric under stress to be more easily grasped.

The experiments are divided into two sets :—

(1) Performed in electric fields.
(2) Performed in varying magnetic fields.

Although these two fields cannot exist separately, their effects are isolated.

SET 1. *Constant Electric Field.*

A constant electric field was produced by two charged metal disks + and −, each of which was stuck on the outside of a glass plate (Plate I. fig. 1). The object of the glass plates was to prevent brush discharges between objects placed in the field and the plates.

Fig. 1, *a*, *b* represents a partially exhausted electrodeless tube, placed in the uniform electric field. In the position shown, the tube is at a uniform potential. Now let it rotate in the direction of the arrow ; the potential of *a* will fall and that of *b* rise until the P.D. between the ends is sufficient to break down the tube. There will then be a convection-current equalizing the potential, during the continuance of which the tube will glow. On rotating still further a similar action will take place.

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* Read November 25, 1892.

The effect of these actions is to produce a double fan-shaped set of images, as shown in fig. 2. The number of images in the fan depends on the intensity of the field.

This phenomenon is intensified by constricting the middle part of the tube.

In designing the tubes for these experiments, care was taken to have the point where they were connected to the pump in such a position that there was no tendency for discharge to take place at these points when in use. The tubes used in this experiment are shown in fig. 3 (*a*, *b*, *c*, and *d*). The stem was used to connect them to the whirling motor.

The surfaces of these tubes should be fairly clean and dry, and it is advisable with a new tube, or one left unused for some time, to excite it between the knobs of an influence-machine.

In the double fan-shaped set of images (see fig. 4, *a*) one end will be seen to be brighter than the other, the current passing from the dull end to the bright end.

Not only is this effect visible with a straight tube, but on rotating an exhausted bulb in the field a stationary luminous region will be seen (see fig. 4, *b*).

Varying Electric Field.

Instead of rotating the tube a similar effect is produced by varying the field in which the tube is placed.

The tube should be placed with its longer axis in the direction of the lines of force. The plates in this case may either be charged and discharged by an influence-machine and dischargers, the latter sparking, or by an induction-coil, with or without sparking.

When an induction-coil is used, very long tubes will glow when brought near.

SET 2. *Magnetic Field.*

The authors were of the opinion that, by taking advantage of the action of the Geissler tube, one might show Hertzian

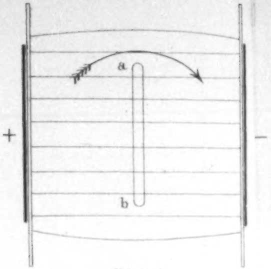


Fig. 1.

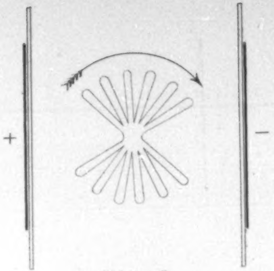


Fig. 2.

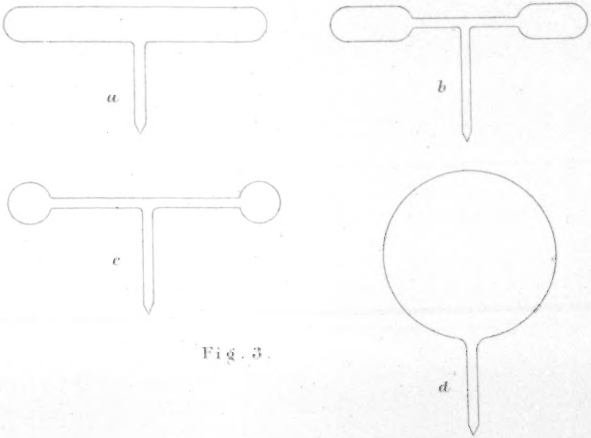
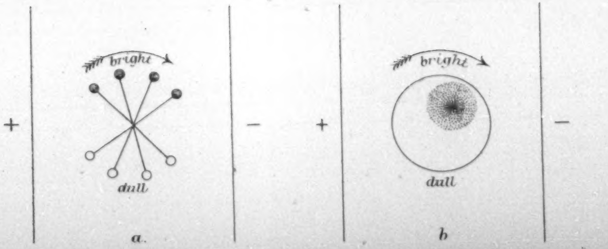


Fig. 3.

Fig. 4.



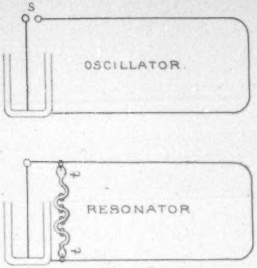


Fig. 5.

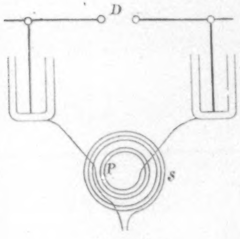
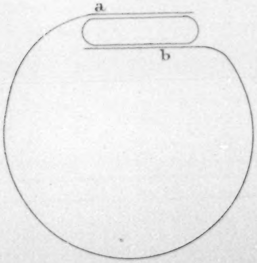


Fig. 7.

Fig. 6.



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phenomena to fairly large audiences. They therefore substituted a Geissler tube for the spark-gap in the resonator.

This was very successful with a Leyden-jar oscillator (fig. 5). The oscillator and resonator consisted of two similar small Leyden jars for capacity, and a single convolution of thick copper wire for the conductor-circuit. The Geissler tube was placed as a shunt to part of the conductor-circuit of the resonator (see fig. 5).

But although they designed special tubes, they were unable to get good results with the ordinary forms of Hertz oscillator and resonator.

They next made a resonator of the following form:—

Two metal plates, *a* and *b* (fig. 6), give capacity, and the wire self-induction. An electrodeless exhausted tube was placed as dielectric between the plates. When a fairly large Hertz oscillator was used, this tube glowed brightly, even when taken to considerable distances.

Glow produced in varying Magnetic Fields.

Instead of, as in the last set of experiments, having a secondary composed partly of a metallic conductor and partly of a vacuum-tube, the whole circuit may be composed of rarefied gas. It is evident that, in order to generate a current in such a circuit, a large induced E.M.F. would be required; this would necessitate either a very strong magnetic field, or one which changed very rapidly. The former was out of the question, the latter was obtained by taking advantage of the phenomenon of oscillatory discharge in the following manner:—

P (fig. 7), a primary consisting of a very few turns of gutta-percha covered wire, was placed in a Leyden-jar circuit, as shown. As a secondary an exhausted bulb, or annular tube, was placed near P.

On sparks passing at D, a bright ring of light appeared in the secondary.

In the actual experiments a metallic secondary, S, was also wound with P and exactly similar to it. On short-circuiting S, the induced current in this circuit will weaken the magnetic field so as to completely extinguish the Geissler secondary.

If S is connected to a third coil this coil will behave in a similar manner to P, with weaker effects; P and S in this case form a one to one transformer.

The wire of the coil P is itself luminous when S is open; short-circuiting S will stop this effect.

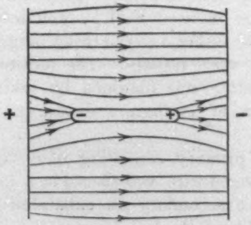
(Since the authors started their researches some of these latter experiments have been shown by Prof. Thomson.)

APPENDIX.

Theory of a Vacuum-tube in a varying Electric Field.

When an electrodeless vacuum-tube is placed axially between two plates, and a potential difference established between them, the tube forms part of the dielectric between the plates, and some of the lines of induction pass through the tube. Now a rarefied gas is a very weak dielectric, and readily gives way under the electric stress; a convection-current flowing through the tube (during which time the gas in the tube behaves as if it were a conductor)*.

Fig. 8.



This results in the ends† of the tube becoming charged, or lines of induction from the plates terminate on the ends of the tube, there being no lines in the tube itself (see fig. 8). The whole of the tube will also be at the same potential.

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* Of course the tube will not conduct after the manner of a copper wire, but rather like a quantity of pith balls between two conducting plates.

† By the term "ends" is meant the *inside* surface of the glass at the ends of the tube.

When the plates are discharged, the lines of induction passing from the plates to the ends of the tube will now pass through the tube; the rarefied gas will again break down under the electric stress, and a convection-current will pass through the tube until the charges at the ends neutralize each other.

If the change in the electric field take place rapidly enough, the convection-current will produce sufficient disturbance of the molecules to cause luminosity.

Theory of a Vacuum-tube rotating in a constant Electric Field.

Consider the tube in a constant field in position *a b* (fig. 9). There will be a − charge at "*a*" and a + one at "*b*", and the tube will behave as a nonconductor. We shall in fact have conditions similar to those in fig. 10, where "*c*" and "*d*" are two insulated conductors, which have been previously connected together by a conductor, and which correspond to the ends of the tube.

Fig. 9.

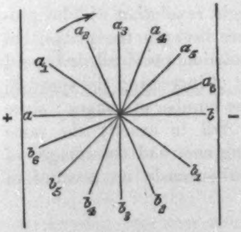
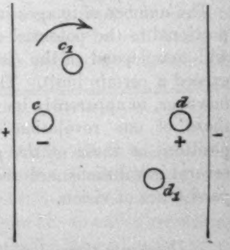


Fig. 10.



Imagine now "*c*" and "*d*" to be moved to positions c_1 and d_1; they will now be no longer at the same potential, and, if connected together, a current will flow from d_1 to c_1, again equalizing their potentials.

The same will happen to the tube if it move during rotation from position $a\,b$ to $a_1\,b_1$; and, when the potential difference between its ends attains a sufficient magnitude, the dielectric of rarefied gas will break down, a convection-current flow from b_1 to a_1, and the tube for the instant will behave like a conductor. During this instant of breaking down the tube will glow. After this the tube will behave like a nonconductor, the ends being at the same potential until the tube rotating moves into position $a_2\,b_2$, and, if the potential difference between a_2 and a_1 is equal to that between a and a_1, and the same for the b's, the tube will glow for an instant at position $a_2\,b_2$.

In the same way it will glow in the positions $a_3\,b_3$, $a_4\,b_4$, &c., if continuously rotated in the direction of the arrow. The points $a_3\,a_4$ &c., $b_3\,b_4$ &c. are such that the potential difference between each consecutive two is approximately the same. As the glow takes place at the end of each alteration of the potential difference, it is obvious that the double fan-like set of images obtained will not be symmetrical with respect to the plates, but will be displaced in the direction of rotation by about one image.

The number of images in a single revolution will be proportional to the potential difference between the plates, but will not depend on the speed of rotation, provided that speed exceed a certain limit. The effect of increasing the speed is, however, to apparently increase the number of images; since those of one revolution do not fall in exactly the same positions as those of the preceding one, and the images of several revolutions are seen simultaneously on account of persistence of vision.

Discharge through Coil surrounding exhausted Bulb.

When a Leyden jar is discharged through a coil of wire, the varying magnetic induction due to the discharge-current will cause an E.M.F. to act in the dielectric inside and outside the ring in a series of concentric circles. Let r be the radius of the coil: consider a circular path in the dielectric of radius x less than r. If the magnetic induction passing through the coil were uniformly distributed, the number of lines passing through the path x would be proportional to its area or to x^2, and the E.M.F. per unit of length along it to x^2 divided by its circumference or to $\frac{x^2}{x}$, *i. e.* to x. Hence the electric stress along a circular path x in the dielectric is greater the larger x. Of course the actual distribution of the magnetic induction is not uniform, but increases as we pass from the centre to the circumference: this will of course make the electric stress increase more rapidly than x. When the dielectric is a rarefied gas it is not broken down until the electric stress exceeds a certain value: hence the discharge takes the form of a luminous ring in the outside of the bulb. By increasing the sparking-distance in discharging the Leyden jar the magnetic induction is increased, making the electric stress greater and broadening the ring.

The breadth of the ring also depends on the degree of exhaustion of the bulb, as the value of the electric stress requisite to break down the rarefied gas depends on the degree of rarefaction. In the case of an annular exhausted tube placed outside the coil the luminous ring will be nearest to the inner side of the tube: since when x is greater than r, the E.M.F. per unit of length is less the greater x, on account of the lines of magnetic induction outside the coil being in the opposite direction through the path x to those inside the coil.

DISCUSSION.

Mr. Swinburne thought some of the effects shown were not Hertzian, but merely cases of ordinary mutual induction. He enquired whether the vacuum-tubes would still glow if the Leyden jars were removed from the so-called resonating circuits. He was also of opinion that in the magnetic experiments the surfaces of the bulb, and not the enclosed gases, took the charges.

Mr. Watson asked if the authors had tried screening off the long waves by a wet cloth. If the effects still existed this would prove that they were Hertzian.

Prof. Ayrton remarked that the only cases where the materials of the bulbs, tubes, &c., did not influence the results were those in which discharges were produced by varying magnetic fields.

Mr. E. T. Carter thought an induction-coil a more efficient machine for producing the glow in tubes than the alternator, &c., used by Mr. Tesla.

Prof. S. P. Thompson said he first noticed that sparks passed between pieces of metal in the vicinity of an induction-coil sparking into a condenser in 1876, when he was showing some experiments on telegraphic apparatus before the Society, but unfortunately he did not pursue the subject. Long before Mr. Tesla's investigations Dr. Bottomley had shown that exhausted tubes could be caused to glow, but it was not until Tesla produced such phenomena on a large scale that people recognized how much light could be got in that way.

Mr. Rimington, in replying to a question by Professor Thompson, said the notes heard when the glass plate approached the condenser were of very high pitch. The explanation, why in the experiments performed in varying magnetic fields the bright parts of the luminous discharges were near the wire, appeared to be that the E.M.F. was greatest in these places. Although he had not tried the experiment suggested by Mr. Swinburne of taking off the Leyden jar, he felt sure that doing so would stop the glow.

III. *The Diffusion of Light.* *By* W. E. SUMPNER, *D.Sc.*[*]

LITTLE information appears to have been published about the diffusing power of unpolished surfaces. The subject has been studied by optical measurements by Zöllner and others, chiefly for astronomical purposes; and the radiation from such surfaces has also been investigated with apparatus designed for measuring radiant heat. But the influence of such diffusion in increasing the illumination of rooms and open spaces, although well known, does not appear to be appreciated to the extent that its importance deserves; and a few numerical determinations of the coefficients of reflexion, absorption, and transmission of diffusing surfaces may prove of interest.

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[*] Read December 9, 1892.

Terms in light are used vaguely, and it will not be deemed out of place to define those which will be here needed. By the *reflecting power* of a surface is meant the ratio of the amount of light reflected by it to the total amount of light incident upon it. The *illumination* of a surface is the amount of incident light per unit area of the surface. The unit *quantity of light* is the flux of radiation per second across a unit area of a sphere of unit radius at whose centre a unit light (of one candle) is placed. The amount of light radiated by a source of k candle-power, within a solid angle Ω, is $k\Omega$, and the total quantity of light emitted by it is $4\pi k$. The *brightness* of a diffusing surface is its candle-power per unit area in the direction normal to the surface. The illumination produced at a point by a surface of brightness B, subtending a (small) solid angle Ω at the point, is $B\Omega$, and if a surface, containing this point, have its normal inclined at an angle ϕ to the axis of Ω, the illumination of this second surface due to the first is $B\Omega \cos \phi$.

If B is the brightness of a surface rendered luminous by reflexion, if η is its (diffused) reflecting-power, and if I is the illumination of the surface, then

$$\pi B = \eta I. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This relation follows from the assumption of the law of cosines, viz. that the candle-power (per unit area) of a bright surface is $B \cos \phi$ in a direction making an angle ϕ with the normal. The right-hand member of (1) is by definition equal to the whole light reflected from unit area of the surface, and must be equal to the integral of $B \cos \phi \, d\Omega$ for all directions on one side of the surface. The value of the integral is easily seen to be πB.

The brightness of a body, as just defined, is directly proportional to the illumination of the image of the body on the retina of the eye, and the word may thus be quite justly used in the ordinary physiological sense of the term. Similarly the law of cosines, just alluded to, is merely another way of expressing the fact that the sensation of the brightness of a diffusing surface is the same from whatever direction this surface may be viewed. For the amount of light received by the eye, and concentrated on the image on the retina, is simply the product of the area of the pupil of the eye and the illumination at the surface of the eye due to the bright object. The former factor is constant for different directions of view, and the latter must vary as the solid angle subtended by the object at the eye, since the area of the image on the retina is a measure of this solid angle. The illumination at a point at which a surface of brightness B subtends a solid angle Ω is not necessarily $B\Omega$ for all inclinations of Ω to the surface, unless the law of cosines is fulfilled; and, if this law is fulfilled, it follows that the illumination of the image of this surface on the retina of the eye is constant at all distances and inclinations. The eye is a good judge, and is indeed the only judge, of quick variations in brightness, and the cosine law is always applicable to diffusing surfaces, the appearance of which does not alter as the eye moves past them. Any divergence from this law is negligible so far as its influence on illumination is concerned.

If Q is the quantity of light radiated per second by the light-sources within a room, and if Q′ is the total amount of light falling on the walls,

$$Q' = Q + \eta Q'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For of the quantity Q′, a portion $\eta Q'$ must be reflected, and the rest absorbed, and the rate at which light is absorbed by the walls must be equal to the rate at which it is produced. The average illumination I′ of the walls of the room must hence be related to I, the illumination due to the direct action of the lights, by the equation

$$I' = \frac{1}{1-\eta} I. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Thus if $\eta = \cdot 90$, $I' = 10\,I$, and if $\eta = \cdot 50$, $I' = 2\,I$, so that the illumination due to the walls of the room may become far more important than that caused by the direct rays of the lights.

The truth of this relation may be also seen as follows:— The light Q falling on the walls is partially reflected, and a quantity ηQ is sent back into the room. This light falls on the walls again and a portion $\eta \times \eta Q$ is reflected a second time. The total quantity of light Q′ falling on the walls owing to successive reflexions is given by the equation*

$$Q' = Q + \eta Q + \eta^2 Q + \ldots = \frac{1}{1-\eta} Q.$$

Or, again, as the illumination I_p' of the walls at any point P is made up of a portion I_p due to the direct rays of the lights, together with a part caused by radiation from the walls, we have

$$I_p' = I_p + \int B \cos\phi \, d\Omega, \quad \ldots \quad (4)$$

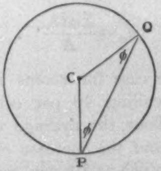
where B is the brightness of the walls, and ϕ is the inclination of the solid angle $d\Omega$ to the normal to the surface at the point P. Assuming that the brightness is the same all over the bounding surface of the room, the value of the integral is readily seen to be πB, and this, as already shown, is equal to $\eta I'$, where I′ is the average illumination of the walls.

When the bounding surface of a room or enclosure consists of portions whose reflective powers are different, the average reflective power may be taken as

$$\eta_m = \frac{\eta_1 A_1 + \eta_2 A_2 + \text{\&c.} \ldots}{A}, \quad \ldots \quad (5)$$

where A is the total area of the bounding surface, of which a portion A_1 has a reflective power equal to η_1, and a second portion A_2 a reflective power η_2, &c. This relation is very approximately true for ordinary rooms, and may be shown to be quite accurate for a spherical enclosure.

Fig. 1.



For let P and Q be any two points of a sphere of centre C and radius r (fig. 1). Then $PQ=2r\cos\phi$, where ϕ is the angle which the chord makes with the radius through either P or Q. Also, with the same notation as before, we have

---

* Since writing this paper I have discovered that this relation has been already pointed out by Mascart [see Palaz, *Traité de Photométrie Industrielle*, p. 268].

$$I_p' = I_p + \int B\cos\phi \frac{dA\cos\phi}{PQ^2},$$

where dA is an element of area at Q of brightness B, and subtending a solid angle $dA\cos\phi/PQ^2$ at P. Now $PQ^2=4r^2\cos^2\phi$ and $\pi B=\eta I'$, where I' is the illumination of the area dA: hence

$$I_p' = I_p + \frac{1}{4\pi r^2}\int I'\eta dA = I_p + \frac{1}{A}\int I'\eta dA, \quad . \quad (6)$$

A being the total area of the spherical surface. The integral is constant whatever the position of the point P, and whatever the character of the reflecting surface of the sphere. Thus if any complete [or if any portion of a] spherical surface be illuminated in *any* manner I_p by the direct rays of a combination of light sources, the actual illumination I_p' will exceed I_p by a constant amount all over the sphere, owing to the reflective action of the surface. Also, if the original distribution be uniform all over the sphere, $I_p=I$, a constant, I_p' will also be constant, $=I'$, and

$$I' = I + \frac{I'}{A}\int \eta dA = I + \eta_m I',$$

where

$$\eta_m = \frac{\Sigma \eta dA}{A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

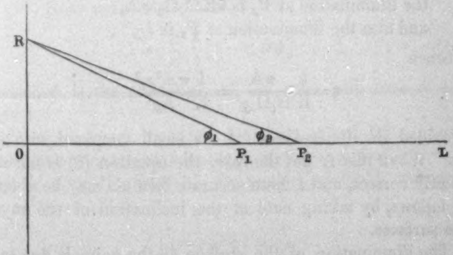
Taking as an average case for rooms a reflecting power of 70 per cent. for the ceiling, 40 per cent. for the walls, and 20 per cent. for the floor, the average value of η is a little over 40 per cent., and the increase of illumination by reflexion becomes as much as 70 per cent. If the walls and ceiling of a room be well whitewashed, the average reflecting power will not fall far short of 80 per cent.; and in such cases the illumination due to diffused reflexion is four times as important as that caused by the direct action of the lights in the room. A further great advantage, resulting from the use of good diffusing surfaces, arises from the fact that the illumination they cause is, in most cases, very approximately constant all over the room, and does not cast shadows.

It is to be noted that when a reflecting surface is coloured, its average reflecting power does not properly represent the character of the increase of illumination caused by it. A room whose walls are covered with red paper whose average reflecting power is 40 per cent., may quite possibly have the red light in the room increased five times owing to the action of the walls. Suppose, for instance, that the reflective power of the paper for red light is 80 per cent., and for other kinds of light only 10 per cent., the average reflecting power will not exceed 40 per cent., yet the red light will be increased five times, while other kinds of light will not be increased to any perceptible extent.

Measurements of Reflective Power.

The surface, whose reflective power was required, was attached to a large screen of black velvet placed at one end O of a 3-metre photometer-bench OL (see fig. 2), so as to

Fig. 2.



be perpendicular to its length. Two lights were used, one being a Methven two-candle gas standard, placed at L, and the other a glow-lamp run generally at about 20 candle-power, placed at P_1. A Lummer-Brodhun photometer was slid along the bench to a point P_2 at which the illumination due to reflexion from the surface OR was equal to that due to the Methven standard at L. At first the lamp P_1 was permanently situated a little to one side of the bench, and the rays from it in the direction of the photometer P_2 were screened off. Subsequently the points P_1 and P_2 were made to coincide, the lamp being fixed to the same slider as the photometer and suitably screened from it and from the eyes of the observer. In some experiments the distance OL was kept fixed and the position of balance OP was determined for each surface tested. In other cases both OP and OL were varied.

Let:—

A = area of diffusing surface on screen OR (centre at O).
Ω_1, Ω_2 = solid angles subtended by A at P_1, P_2 respectively.
x_1, x_2 = distances OP_1, OP_2 respectively.
y = distance P_2L.
K = candle-power of the glow-lamp at P_1.
k = candle-power of Methven standard at L.

Then:—

The quantity of light falling on the area A is $K\Omega_1$,
the average illumination of A is $K\Omega_1/A$,
the average brightness is $\eta K\Omega_1/\pi A$, by (1),
the illumination at P_2 is $\eta K\Omega_1\Omega_2/\pi A$,
and also the illumination at P_2 is k/y^2;

whence

$$\eta = \frac{k}{K}\frac{\pi A}{\Omega_1\Omega_2 y^2} = \frac{k}{K}\frac{\pi x_1^2 x_2^2}{A y^2}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

provided the dimensions of A are small compared with x_1 or x_2. When this is not the case, the equation (8) is not sufficiently correct, and a more accurate formula may be obtained as follows, by taking note of the inclination of the rays to the surfaces.

The illumination of the surface at the point R due to the lamp at P_1 can be easily shown to be $K\cos^3\phi_1/x_1^2$, where ϕ_1 is the angle between RP_1 and OL, the line of centres of the bench. The brightness B of the surface at R is therefore $\eta K\cos^3\phi_1/\pi x_1^2$ by equation (1). An element of area dA at R subtends at P_2 a solid angle $d\Omega_2$ equal to $\cos^3\phi_2 dA/x_2^2$, and the illumination at P_2 due to this element is $Bd\Omega_2 = \eta K\cos^3\phi_1\cos^3\phi_2 dA/\pi x_1^2 x_2^2$ on any area placed perpendicular to RP_2. As the photometer screen is perpendicular to the bench and not to RP_2, we must multiply this expression by $\cos\phi_2$ to get the effective illumination due to the element dA. Finally we have for the total illumination the integral:—

$$I = \frac{\eta K}{\pi x_1^2 x_2^2} \int \cos^3\phi_1 \cos^4\phi_2 \, dA, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

in which the angles ϕ_1, ϕ_2 are related by the equation

$$x_1 \tan\phi_1 = x_2 \tan\phi_2.$$

When the area A is circular, with O as its centre, this integral reduces to

$$I = \frac{\eta K}{x_1^2} \int \cos^3\phi_1 d \, . \sin^2\phi_2. \quad . \quad . \quad . \quad (10)$$

The value of this integral can be readily evaluated, but it does not lead to a convenient formula, and as it was found practically preferable to fix the lamp to the same slider as the photometer, and at the same distance from the screen OR, we may put

$$\phi_1 = \phi_2 = \phi,$$

$$x_1 = x_2 = x,$$

and (10) then reduces to

$$I = \eta \frac{K}{x^2} \frac{2}{5} [1 - \cos^5\phi], \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

in which ϕ is the semiangle of the cone with base A and height x.

This expression is rendered more convenient for purposes of calculation by taking advantage of the fact that $A/\pi x^2$, or $\tan^2\phi$, is a small quantity. By neglecting $\tan^6\phi$ compared with unity we obtain

$$\frac{2}{5} [1 - \cos^5\phi] = \frac{1}{X},$$

where

$$X = \frac{\pi x^2}{A} + 1{\cdot}75 + {\cdot}43 \frac{A}{\pi x^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and in most cases it will be found that the third term in this expression is negligibly small compared with the sum of the other two.

The value of I found in (11) may be equated to k/y^2 when the photometer is in the position of balance, and on doing so, we find for η the value

$$\eta=\frac{k}{K}\frac{x^2}{y^2}X, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

which reduces to (8) when x is large.

This equation was used for the great majority of the reflexion tests. With feebly diffusing surfaces such as black cloth, the distance x had to be so much reduced that the area A subtended a greater angle at the photometer-screen than the aperture of the photometer itself. In such cases the angle ϕ in (11) was calculated from the solid angle subtended by the aperture of the photometer at the centre of the photometer-screen. With diffusing surfaces which appeared to shine slightly under the action of light, the effect of regular reflexion had to be separated from that of diffused reflexion, by experiments made with the same surface for different values of x. In such cases, which will be alluded to subsequently, the true reflecting-power is not given by formula (13). The areas A of the reflecting surfaces used in the experiments were never circular, as assumed in the above proof, but as they were always approximately square, with the central portion at O (fig. 2), any error in (12) and (13) arising in this way must be quite negligible. The ratio k/K of the two lights was frequently tested in the course of the experiments and was found very constant during every set of tests.

The results obtained are given in the accompanying table. In the majority of cases the numbers given are approximate only, as there seemed no object in aiming at great accuracy. The first four surfaces referred to in the table, viz., thick white blotting-paper, white (rough) cartridge-paper, tracing-paper, and tracing-cloth, were, however, carefully tested, and the numbers obtained represent the mean of many observations.

Zöllner found white surfaces to reflect from 70 to 78 per cent. of the light incident upon them. The numbers given at the head of Table I. are slightly higher. They, however, agree very well with results which have been published with regard to the diffuse reflexion of radiant heat, since 82 per cent. is generally given as the reflective power of white substances*.

TABLE I.—Reflecting-Powers.

	per cent.		per cent.
White blotting-paper	82	Plane deal (clean)	40 to 50
White cartridge-paper	80	,, (dirty)	20
Tracing-cloth	35	Yellow cardboard	30
Tracing-paper	22	Parchment (one thickness)	22
Ordinary foolscap	70	,, (two thicknesses)	35
Newspapers	50 to 70	Yellow painted wall (dirty)	20
Tissue-paper (one thickness)	40	,, ,, ,, (clean)	40
,, (two thicknesses)	55	Black cloth	1·2
Yellow wall-paper	40	Black velvet	0·4
Blue paper	25		
Dark brown paper	13		
Deep chocolate paper	4		

The degree of consistency of the results may be judged as follows:—In a series of 10 determinations of the reflective power of white blotting-paper, made with values of x (see fig. 2), varying from 40 cm. to 82 cm., the mean reflecting-power was found to be 82·4 per cent., and the average error of a single determination from the mean was 1·4. With surfaces of lower reflecting-power (such as tracing-cloth) the numbers obtained in successive experiments were more consistent. The value of η for white blotting-paper was checked by comparing it directly with that of a piece of common mirror. The Methven standard at L (see fig. 2) was replaced by the mirror, arranged so as to reflect the light from the glow-lamp along the line of the bench. The reflective power of the white blotting-paper placed on the screen OR was found to be 98·5 per cent. of that of the mirror. The value of η for the mirror, for normal rays, was separately determined and found to be 82 per cent., and hence η for the blotting-paper comes out as 80·8 per cent.

Several of the numbers in the above table were confirmed by comparative measurements, using white paper as a standard reflector, the Methven standard at L (fig. 2) being replaced by a surface of white paper exposed to the rays of the glow-lamp at P. The reflective power of one of the walls tested was measured in diffuse daylight by exposing the aperture of a photometer to the radiation of the wall, and balancing the illumination against that of a standard candle. The part of the wall affecting the photometer was then covered with a sheet of white blotting-paper, and the ratio of the two illuminations at once gave the ratio of the reflecting-powers.

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* See Jamin et Bouty, Tome Troisième, p. 149. Results by MM. Goddard, De la Provostaye et Desains.

Measurement of Absorption.

The absorbing-powers of some of the preceding substances were determined by measuring the candle-power of the light from a glow-lamp, first when this was uncovered, and afterwards when it was surrounded with a cylinder of the paper under test. The cylinders were short, being just longer than the height of the lamp, and were closed at top and bottom with caps of the same paper, so that the lamp was completely enveloped by the paper tested. The ratio of the diminution of candle-power to the original candle-power gave the *apparent* absorption of the paper. If the candle-power in some particular case was found to diminish 30 per cent., it would have been erroneous to conclude that the envelope absorbed as much as 30 per cent. of the light incident upon it, or that 70 per cent. was transmitted. If the reflecting, absorbing, and transmitting powers of a material be respectively denoted by η, a, and τ, there must exist between these quantities the relation

$$\eta + a + \tau = 1.$$

Also if Q be the quantity of light given out per second by the light-source within the envelope, the quantity of light incident per second upon the surface of the envelope will, owing to internal reflexion, be increased to Q′, where

$$Q'(1-\eta) = Q;$$

the quantities of light absorbed and transmitted will respectively be

$$aQ' \text{ and } \tau Q',$$

and the ratios these quantities bear to Q will similarly be

$$\frac{\alpha}{1-\eta} \text{ and } \frac{\tau}{1-\eta};$$

the sum of course being unity.

If the light-source can be assumed to radiate equally in all directions, the ratio of its candle-power, after putting on the envelope, to the original candle-power will not be τ but $\tau/(1-\eta)$. The influence of internal reflexion is to increase *both* the absorption and the transmission, and unless it is taken into account large errors may be made in estimating the coefficients. A very simple way of showing the effect of internal reflexion consists in surrounding a glow-lamp with a white paper cylinder, open at the top, and adjusting a photometer screen till the illumination caused by it is balanced against that of some standard source of light. If now a piece of white paper be placed on the top of the cylinder, so as to shut in the vertical rays of the lamp, the candle-power in the horizontal direction will be found to increase considerably. In some of the tests, in order to avoid error caused by non-uniform radiation of the lamp, this was first surrounded with an envelope of tracing-cloth, or blotting-paper, and the combination used as the light-source. If k_0 be its candle-power in this condition, and if k_1 be the observed candle-power after completely surrounding it with an envelope of the substance to be tested, the apparent absorption of this material is

$$\frac{k_0-k_1}{k_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and the true absorption coefficient α is given by

$$\alpha=(1-\eta)\,\frac{k_0-k_1}{k_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where η is the previously found reflexion coefficient.

When an inner envelope is used to produce a light which radiates equally well in all directions, it is necessary for the outer envelope to be large compared with it, since otherwise the formula for the increase of illumination due to internal reflexion cannot be applied.

The following table gives the values of the absorbing coefficients, expressed as percentages, for the four substances at the head of Table I., from which the corresponding values of η used in applying (15) have been taken.

TABLE II.

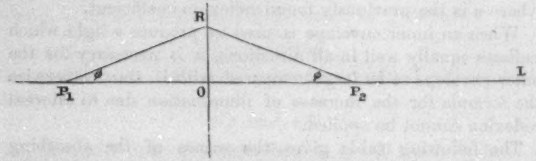
	Apparent Absorption.	Real Absorption.
White Blotting-paper	77 per cent.	13·8 per cent.
White Cartridge-paper	61 „	12·2 „
Tracing-cloth	23 „	15·0 „
Tracing-paper	9 „	7·0 „

Three large glass globes, made for arc-lamps, were also tested. As the reflective powers of the globes could not easily be found, only the apparent absorption was measured. One globe was of opal glass (almost transparent), and the ratio of apparent absorption was 15 per cent. A second globe was of ground glass and absorbed 42 per cent. The remaining one was of opal glass, too opaque to allow any bright object placed within it to be distinguished. This absorbed 39 per cent.

Measurements of Transmitting-Power.

The amount of light transmitted through the surfaces above mentioned was measured in a very similar manner to that in which their reflective powers were determined; the only difference being that the glow-lamp at P_1 (see fig. 2) was moved to the opposite side of the surface as in fig. 3.

Fig. 3.



$OP_1 = OP_2 = x. \quad P_2L = y.$

OR represents the screen of paper, pinned on a wooden frame, and placed perpendicular to the optical bench P_1L. The glow-lamp was at P_1, the photometer at P_2, and the Methven standard at L. The distances OP_1, OP_2 were arranged to be equal, and balance obtained by moving L along the bench. On referring to equations (8) to (13) and the arguments used in establishing them, it will be seen that they are all applicable to the case now considered if only we substitute τ, the transmitting-power, for η.

When, however, tests were made with the paper surfaces already referred to, it was soon found that the numbers calculated from expression (13),

$$\frac{k}{K}\frac{x^2}{y^2}X=Y, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

were not constant for the same substance. They differed from each other far more than could be accounted for by errors of experiment; thus, the values found for τ by this formula were too high, and frequently exceeded 100 per cent. For any given surface the values were found to increase progressively with the value of x used in the tests. The reason for this is easily seen when it is remembered that equations (8) to (13) are only true on the assumption that the surfaces considered are purely diffusive, and do not alter in appearance as the eye changes its point of view. When light is transmitted through a semitransparent substance such as tracing-paper, or tracing-cloth, the brightest part of the surface is always on the line joining the eye to the light, and visibly moves over the surface as the point of view is changed. The easiest way to represent these facts is to assume that, of the light transmitted, a portion τ_1 passes through without change of direction, and that the rest τ_2 is diffused in accordance with the cosine law. The case is analogous with a reflecting surface such as white enamelled iron, which reflects a portion η_1 of the incident light in accordance with the regular law of reflexion, and diffuses another portion η_2 according to the law of cosines. On referring to equations (8) to (11) it will be noticed that they are still true for the illumination due to diffusion if we substitute for η either τ_2 or η_2 (according as we are considering transmission (fig. 3) or reflexion (fig. 2) respectively). The additional illumination at the photometer due to regular, *i. e.* direct, transmission (transparency) is

$$\tau_1 \frac{K}{(2x)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and a similar expression holds for regular reflexion if we substitute η_1 for τ_1.

The whole illumination must, as before, be equal to k/y^2, and by (11), (12), and (17) we have

$$\tau_2 \frac{K}{x^2 X} + \tau_1 \frac{K}{4x^2} = \frac{k}{y^2}, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

whence the value of Y in (16) is equal to

$$Y = \tau_2 + \frac{\tau_1}{4} X ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and a similar expression holds for reflexion if we replace τ_1 and τ_2 by η_1 and η_2 respectively. The true values for the transmitting and reflecting coefficients are

$$\left.\begin{array}{l} \tau = \tau_1 + \tau_2, \\ \eta = \eta_1 + \eta_2, \end{array}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and the reason the values found for Y were too high, and became greater and greater as x increased, was simply that the values used for X (see 12) were always greater than 4, and increased rapidly with x.

By plotting the numbers found for Y with the corresponding values of X, a straight line is obtained from which the values represented by the symbols in (20) can all be determined. The straightness of these lines, and the verification of the fundamental formula

$$\eta + \alpha + \tau = 1,$$

affords a good criterion of the extent to which the principles and the formulæ, referred to in this paper, can be relied upon.

The following Tables III., IV., and V. contain the results of three sets of tests on the transmitting-power of blotting-paper and tracing-cloth, and on the reflecting-power of tracing-cloth. The values of $Y_{\text{obs.}}$ are calculated with the aid of (12) and (16) from the observed values of x and y. All the dimensions are given in centimetre-units. In the accompanying sheet of curves the values of $Y_{\text{obs.}}$ are plotted as ordinates, with the corresponding values of X as abscissæ. From the straight line most nearly representing the connexion between the points the values of $Y_{\text{calc.}}$ have been obtained, and are given in the tables. From this straight line also the true value of the transmitting (or reflective) power can be found by finding the value of the ordinate when $X=4$. The intercept on the axis of Y shows the portion of the light which is diffused.

Table III.—Transmitting-Power of Blotting-paper.

$A=980$. $K=26$. $k=2$. $\tau=9·2$. $Y_{\text{calc.}}=6·50+2·7\,X/4$.

x	60	50	40	45	55	60
y	154	120	88·4	104·6	137	154
X	13·6	9·8	7·0	8·25	11·40	13·6
$Y_{\text{obs.}}$	15·5	13·2	11·2	12·0	14·4	15·5
$Y_{\text{calc.}}$	15·7	13·1	11·2	12·1	14·2	15·7

Table IV.—Transmitting-Power of Tracing-cloth.

$A=980$. $K=23·1$. $k=2$. $\tau=54·4$. $Y_{\text{calc.}}=41·0+13·4\,X/4$.

x	40	50	60	70	80	75	65	55	45
y	39·2	53·5	70·1	89·4	105·6	95·8	77·7	61	46·1
X	6·98	9·80	13·6	17·4	22·30	19·4	15·4	11·2	8·25
$Y_{\text{obs.}}$	63·5	74·3	86·1	97·0	116	107	91·8	81·7	69·3
$Y_{\text{calc.}}$	64·4	73·7	85·6	99·2	115·8	106	94·0	78·5	68·7

Table V.—Reflecting-Power of Tracing-cloth (shiny side).

$A=1300$. $K=12$. $k=2$. $\eta=35·7$. $Y_{\text{calc.}}=30·9+4·8\,X/4$.

x	50	55	62·3	59·8	47·5	44·5
y	90	104·4	127·7	120·2	82·5	75·5
X	7·82	9·10	11·13	10·45	7·22	6·55
$Y_{\text{obs.}}$	40·4	41·8	44·1	43·2	39·6	38·6
$Y_{\text{calc.}}$	40·3	41·8	44·2	43·4	39·6	38·7

Several sets of tests were taken. Some of these are represented by the sheet of curves shown in fig. 4. Y is the percentage of the incident light which the surface *apparently* reflects or transmits, assuming that it is all diffused, and is calculated from equation (16). The true values of the coefficients are obtained from the curves by applying equations (19) and (20). The numerical details of the observations are of no special interest, and the essential results may be summarized as follows :—

Fig. 4.

TABLE VI.—Reflective Powers.

	Regular. η_1.	Diffused. η_2.	Total. η.
Blotting-paper	0 per cent.	82 per cent.	82 per cent.
Cartridge-paper	0 „	80 „	80 „
Tracing-cloth (shiny side) ...	4·8 „	30·9 „	35·7 „
„ „ (rough side)...	2·7 „	31·6 „	34·3 „
Tracing-paper	2·2 „	19·8 „	22·0 „

TABLE VII.—Transmissive Powers.

	Direct. τ_1.	Diffused. τ_2.	Total. τ.
Blotting-paper	2·7 per cent.	6·5 per cent.	9·2 per cent.
Cartridge-paper	2·5 ,,	8·7 ,,	11·2 ,,
Tracing-cloth	13·4 ,,	41·0 ,,	54·4 ,,
Tracing-paper	29·8 ,,	46·2 ,,	76·0 ,,

We may now collect the coefficients η, a, τ, determined by the foregoing independent methods, and compare their sum with unity.

TABLE VIII.

	η.	a.	τ.	$\eta+a+\tau$.
Blotting-paper	82 per cent.	13·8 per cent.	9·2 per cent.	105·0 per cent.
Cartridge-paper ...	80 ,,	12·2 ,,	11·2 ,,	103·4 ,,
Tracing-cloth	35 ,,	15·0 ,,	54·4 ,,	104·4 ,,
Tracing-paper	22 ,,	7·0 ,,	76·0 ,,	105·0 ,,

The numbers in the last column differ from the true value of 100 per cent. to a greater extent than can fairly be accounted for by the limits of experimental error. They are all over 100; and this was the case, not only for the tests here given, but also for every one of many sets of tests taken. The small discrepancy would be accounted for by assuming that the law of cosines is not exactly fulfilled. A very slight departure from this law would be amply sufficient to explain the results.

Suppose the candle-power of a unit area of a diffusing surface in the direction of the normal is B, and in any direction ϕ is

$$B\,(\cos\phi)^{1+\epsilon}.$$

It is then easy to show that the total amount of light given out per unit area is equal to

$$\pi B\,\frac{1}{1+\frac{\epsilon}{2}}.$$

In all the above experiments the actual measurements have referred to the light receding from the diffusing substance at inclinations all practically normal to its surface. The quantity B has been measured, the total light has been calculated as πB, and has been overestimated in the ratio $1+\frac{\epsilon}{2}:1$.

To account for an error of 5 per cent., the quantity ϵ need only be 0·1. The brightness of the surface (in the physiological sense) would, when viewed at an inclination ϕ, be proportional to

$$B\,(\cos\phi)^{1+\epsilon}/\cos\phi, \quad \text{or} \quad B\,(\cos\phi)^{\epsilon}.$$

This quantity is practically constant (if $\epsilon=0{\cdot}1$) until ϕ becomes very large. Its value is ·994 for $\phi=20^\circ$ and ·974 for $\phi=40^\circ$, and the change in the brightness of the surface would hardly be perceptible to the eye.

This correction applies to the coefficients η_2 and τ_2; it does not affect the values of η_1, α, or τ_1. In some of the first tests of reflecting-power the inclination of the light-rays to the surface was considerably less than 90 degrees, and the values of η obtained were less than those given above. These tests confirm the idea that the cosine law is not strictly fulfilled, but they were not accurate enough to be conclusive.

The above measurements were all made in the Optical Laboratory of the Central Institution, and the writer has had the benefit of the assistance of some of the students of that College in re-testing and confirming the results given in the foregoing tables.

DISCUSSION.

Mr. A. P. Trotter said he had been interested in the subject of diffusion for many years with a view to obviating the glare of arc lamps. The reflecting-power of substances was of great importance in the illumination of rooms; in one case, measured by Dr. Sumpner and himself, two-thirds of the total illumination was due to the walls. Referring to the cosine law, he said he had found it true except when the angles of incidence approached 90°. In cases where considerable total reflexion took place, the apparent brightness near the normal direction was greatly in excess of that in other directions. These points he illustrated by polar curves. He had also considered what should be the nature of a roughened or grooved surface to give the cosine law of diffusion. No simple geometrical form of corrugations, &c., seemed to fulfil the required conditions.

Dr. Hoffert said the high numbers given for the reflecting-powers of substances were very interesting. Most people had noticed the effect of laying a white table-cloth in an ordinary room. He had also observed that wall-papers of the same pattern, but slightly different in colour, had very different effects in producing increased illumination, and wished to know if the influence of small differences in colour and texture on diffusing-power had been investigated.

Mr. Blakesley defended the cosine law, and suggested that the summation of the powers exceeding unity might be due to the fact that the enclosure reflected heat as well as light, thus raising the temperature and increasing the efficiency of the radiant.

Dr. C. V. Burton did not understand why the cosine law should be objected to, for it was possible that no surface was perfectly diffusive. The effect of reflexion from walls, &c., say in illuminating a book, would not, he thought, be so great as would appear from the numbers given, for one usually read near a light and the reflected light falling on the book was only a small part of the whole, on account of the greater distances of the walls.

Dr. Sumpner in reply said he had, as stated in the paper, used white blotting-paper as a standard of reflecting-power and found it very convenient. His most careful measurements had been made on whitish surfaces and not on coloured ones. Where one colour, say red, preponderates in a room, the average light would be much redder than that emitted by the source, owing to the other colours being absorbed. In considering illumination as related to distinct vision, it was necessary to take account of the eye itself, for the pupil contracted in strong lights and opened in feeble ones. This subject he hoped to treat fully in a subsequent paper.

IV. *The Laws of Molecular Force.*

By WILLIAM SUTHERLAND, *M.A.*, *B.Sc.**

IN my last paper on this subject (Phil. Mag. April 1889) it was shown that if the law of force between two similar molecules is $3Am^2/r^4$, the parameter A could be calculated for a large number of bodies from Robert Schiff's measurements of their surface-tensions, and a law was announced connecting the values of A with chemical composition; but as this law was affected with exceptions in the case of the organic bromides and iodides and the amines, and as the argument from surface-tension requires the introduction of a considerable number of assumptions, I felt that it was very desirable to secure some other means of finding A. It soon appeared that the only satisfactory plan would be to return again to the search for the true characteristic equation of fluids on the model of Clausius's virial equation, as I had found that not one of those hitherto advanced was capable of general application. Fortunately the experimental material now available is so abundant and so well placed that I was completely successful in the quest, in the more tedious parts of which I had the advantage of assistance from my brother, Mr. John Sutherland. Amagat's exhaustive study of the compression and expansion of gases, along with Ramsay and Young's work on ethyl oxide, were the groundwork of the research.

As the true characteristic equations for fluids are the key to many chambers of molecular physics, I have been able to enter many of these, and especially to discover the true law of the parameter A without exception. To give an idea of the scope of these investigations free from detail, I furnish the following table of contents:—

1. Establishment of characteristic equation for compounds above the region of the critical volume, with proof that there is discontinuity in the liquefaction of compounds.

2. The same for the gaseous elements, with proof of continuity during liquefaction in their case.

3. Brief discussion of exceptional compounds such as the alcohols and ethylene.

4. Establishment of characteristic equation below the region of the critical volume, its main feature being the occurrence

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* Read June 24, 1892.

in it of the same internal virial constant as in the equation for the region above the critical volume.

5. A short digression on the general interpretation of Clausius's equation of the virial.

6. Consideration of Van der Waals's generalization that if each substance has its temperature, pressure, and volume expressed in terms of the critical values as units, one and the same law applies to all bodies; proof that this is true for elements and compounds separately, above the critical region, and approximately true below.

7. Five methods of finding the internal virial constant: first, from the expansion and compression of the substance as gas or vapour; second, from its expansion and compression as liquid; third, from its latent heat; fourth, from its critical temperature and pressure; fifth, from its surface-tension: all the methods being afterwards proved to give accordant results for a large number of compounds.

8. The fifth or capillary method treated in greater detail, with digressions on the Brownian movement and molecular distances.

9. Establishment on theoretical grounds of Eötvös's relation between surface-tension and molecular domain (volume).

10. Tabulation for a large number of bodies of the product of the square of the molecular mass by the virial constant determined by several methods, and verification thereby of the general principles preceding.

11. Establishment of the law connecting the virial constant of a substance with its chemical composition. Definition of the Dynic Equivalent of a substance and determination of its value for several radicals.

12. Close parallelism between Dynic Equivalents and Molecular Refractions.

13. Return to the discontinuity during liquefaction of compounds, and proof that it is due to the pairing of molecules.

14. Brief discussion of the constitution of the alcohols as liquids.

15. Methods of finding the virial constant for inorganic compounds, including a theory of the capillarity and compressibity of solutions.

16. Tabulation of the product of the square of the molecular mass by the virial constant for inorganic compounds, and determination of the Dynic Equivalents of the metals in the

combined state. Again a close parallelism between dynic equivalents and molecular refractions or refraction-equivalents.

17. Meaning of this parallelism; general speculations as to the volumes of the atoms and their relation to ionic speeds.

18. Attempt to determine the velocity of light through the substance of the water-molecule.

19. Suggested relation between the change in the volume of an atom on combination and the change in its chemical energy.

1. *Establishment of the characteristic Equation for Compounds above the region of the Critical Volume*, with proof that there is discontinuity in the liquefaction of compounds.—Amagat established (*Ann. de Chim. et de Phys.* sér. 5, t. xxii.) that for gases $\partial p/\partial T$ is a function of volume only down to volumes near the critical, but that at lower volumes it begins to vary with temperature. Ramsay and Young (Phil. Mag. May 1887), while verifying the independence of $\partial p/\partial T$ on temperature above the critical volume for such bodies as ethyl oxide and the alcohols, sought to show that this independence continues right into the liquid state; but, as a matter of fact, their temperature-range in the experiments below the critical volume is not great enough to decide the question one way or another. We shall see that in the case of compounds Amagat's conclusion is the correct one, while in the case of the elements Ramsay and Young's contention appears to hold. The conviction that $\partial p/\partial T$ becomes slightly variable with temperature below the critical volume was one reason that determined me to represent the behaviour of fluids by two equations merging into one another; the one applying down to near the critical volume, the other below that.

Clausius's equation of the virial is to be our guide in studying $\partial p/\partial T$,

$$\tfrac{3}{2}pv=\Sigma\tfrac{1}{2}mV^2-\tfrac{1}{2}\cdot\tfrac{1}{2}\Sigma\Sigma Rr,$$

where V is velocity and R is force between two molecules at distance r apart.

According to the law of the inverse fourth power, the double sum of the internal virial reduces to an expression varying inversely as the volume and independent of the temperature, as I have shown before (Phil. Mag. July 1887).

If we integrate Amagat's relation $\partial p/\partial T=f(v)$, we get

$$p=f(v)T+\phi(v);$$

and as in the perfect gaseous state $pv=\mathrm{RT}$, where v being the volume of unit mass and T reckoned as temperature C. + 273°, R varies inversely as the molecular weight of the substance, we will write our equation in the form

$$pv=\mathrm{RT}vf(v)+v\phi(v),$$

where we see that $v\phi(v)$ stands for the internal virial-term. If, then, according to the law of the inverse fourth power, the internal virial varies inversely as v, then $v^2\phi(v)$ ought to be constant. Now Ramsay and Young have carefully tabulated the values of $\mathrm{R}f(v)$ and $\phi(v)$ for different values of v in the case of ethyl oxide; so it is easy to tabulate $\mathrm{R}vf(v)$ and $v^2\phi(v)$, as we proceed to do in the following Table, where v is the volume of a gramme of ethyl oxide in cubic centimetres, and the metre of mercury is the unit of pressure. These units will be used throughout when we are dealing with experimental results involving pressures; but when necessary, for theoretical convenience, we will convert to absolute units.

Table I.—Ethyl Oxide.

v.	$\mathrm{R}vf(v)$.	$v^2\phi(v)$.	v.	$\mathrm{R}vf(v)$.	$v^2\phi(v)$.
Perfect gas.	·842		3·3	1·758	2413
100	·912	5710	3·0	1·865	2366
50	·973	5190	2·75	2·013	2371
20	1·129	4554	2·5	2·30	2487
15	1·201	4302	2·4	2·42	2519
10	1·327	3908	2·3	2·56	2550
8	1·409	3661	2·2	2·73	2589
6	1·525	3308	2·1	2·95	2646
5	1·595	3047	2·0	3·19	2691
4	1·654	2656	1·9	3·54	2771
3·7	1·682	2534			

The critical volume of ethyl oxide is between 5 and 4; so that if $\partial p/\partial \mathrm{T}$ does become variable with temperature below the critical volume, the values of $\mathrm{R}vf(v)$ and $v^2\phi(v)$, calculated for volumes below 5, on Ramsay and Young's assumption that even below the critical volume $\partial p/\partial \mathrm{T}$ is independent of temperature, will be affected with an error of more or less importance; they may therefore be regarded as a first approximation only and are added for comparison.

The first point to notice in these numbers is that $\mathrm{R}vf(v)$ increases steadily from its limiting value ·842 in the perfect gas state to double that amount near the critical volume, while at the same time $v^2\phi(v)$ diminishes from its limiting value in the gaseous state to the half of it near the critical volume. This result would seem at once to contradict the law of the inverse fourth power; but we shall see in the sequel that, in compression down to the critical region, there is a process of pairing going on among the molecules and producing this departure from the requirements of the law of the inverse fourth power, uncomplicated by such a process.

It is to be noted that the limiting value of $v^2\phi(v)$ is difficult to determine experimentally, because $\phi(v)$, the quantity measured, tends to the limit zero. But while below volume 4, $\mathrm{R}vf(v)$ increases with increasing rapidity, $v^2\phi(v)$ remains almost stationary, it dips a little and then increases; but remembering that its values count only as first approximations, we may assume that $v^2\phi(v)$ attains near the critical volume a value which remains constant in the liquid state, and is about half of the limiting value for the gaseous state. Thus there is discontinuity in the passage from the region above the critical volume to that below (or, more briefly but less accurately, during liquefaction). We must note carefully that in the range of volume from 4 to 1·9, which is a large liquid range, $v^2\phi(v)$ remains constant, as it should according to the law of the inverse fourth power, now that the process of pairing is completed.

To represent $\mathrm{R}vf(v)$ I found the form $\mathrm{R}\{1+2k/(v+k)\}$ to be efficient; it gives the limit 2R to the function when $v=k$; and as ·842 is the known value of R, each of the above tabulated values of $\mathrm{R}vf(v)$ yields a value of k, the mean value 4·066 having been adopted by me. The other function, $v^2\phi(v)$, proved no less amenable to simple representation, the form found to fit it being $lv/(v+k)$, which attains the value $l/2$ when $v=k$; and as the value of k is known, we can calculate from each tabulated value of $v^2\phi(v)$ a value of l, and again adopt the mean value 5514. Hence down to near the critical volume we have the behaviour of ethyl oxide represented by the simple form of equation

$$pv=\mathrm{RT}\left(1+\frac{2k}{v+k}\right)-\frac{l}{v+k}, \text{ with } k=4{\cdot}066 \text{ and } l=5514,$$

involving only the two constants k and l peculiar to ethyl oxide.

We will now compare a few values of the pressures, in metres of mercury, given by this equation with those found by Ramsay and Young.

TABLE II.

	Volume	100.	50.	20.	10.	6.
175° C.	Pressure, experiment.	3·500	6·620	13·88		
	Pressure, calculated.	3·538	6·634	13·78		
195° C	Pressure, experiment.	3·710	7·020	15·06	23·00	27·00
	Pressure, calculated.	3·719	7·021	14·90	22·99	27·44
280° C.	Pressure, experiment.	......	......	19·80	34·59	49·62
	Pressure, calculated.	......	......	19·69	34·28	49·00

The agreement is as close as can be looked for; because although Ramsay and Young measure volumes to within ·01 centim., and pressures to within 2 centim. of mercury, the quantities measured cannot be considered known with that degree of accuracy; for the discrepancies between their measurements and those of Perot (*Ann. de Chim. et de Phys.* ser. 6, t. xiii.), who made special determinations in a large globe of the saturation-volume of ethyl oxide at different temperatures, are greater than those in Table II. Accordingly it would be useless to seek a better empirical representation of Ramsay and Young's results than the above; and as we are chiefly interested in establishing our simple form of characteristic equation, we had better proceed at once to the consideration of Amagat's experiments on carbonic dioxide, practically identical with Andrews's, but more extensive. Amagat's unit of volume is $\frac{1}{3410}$ of that occupied by the gas at 0° and 1 atmosphere; taking the weight of a litre of the gas at 0° and 1 atmosphere as 1·9777 gramme, we can convert Amagat's data to the gramme and centim. as units. The following Table is arranged in the same way as Table I.:—

TABLE III.—Carbonic Dioxide.

v.	$Rvf(v)$.	$v^2\phi(v)$.	v.	$Rvf(v)$.	$v^2\phi(v)$.
Perfect gas.	1·421		2·64	2·56	1661
7·34	1·92	2159	2·35	2·64	1566
(7·00)	(2·11)	(2500)	2·05	2·77	1482
5·14	2·13	2071	1·91	2·86	1441
(5·00)	(2·37)	(2400)	1·76	3·02	1430
3·67	2·31	1858			

(The bracketed numbers are introduced from Andrews.)

A glance at this Table shows the same facts to be in it as in Table I. The critical volume of CO_2 is somewhere about 2; and we notice that near this volume $Rvf(v)$ tends to double the value 1·421 in the gaseous state, while at the same point $v^2\phi(v)$ approaches a constant value about the half of what must from inspection be estimated as the upper limit of it. Both functions are accurately represented by the same forms as in the case of ethyl oxide with $k=1{\cdot}762$ and $l=2773$. With these values the following pressures were calculated for comparison with experiment:—

TABLE IV.—Carbonic Dioxide.

	Volume	11·74.	8·8.	5·87.	3·67.	2·64.	2·20.
100° C.	Pressure, experiment.	39	49·8	69	96·8	124	143
	Pressure, calculated.	39·4	50·4	70	99	123	137
70° C.	Pressure, experiment.	34·5	43·7	58·8	79·5	95	105
	Pressure, calculated.	34·8	44·0	59·4	80·0	94	100
35° C.	Pressure, experiment.	29·7	36·5	46·4	55·8	61	63·6
	Pressure, calculated.	29·5	36·5	47·0	57·5	60	57·7

The agreement is quite satisfactory except at the lowest volume, which is near the critical; and I have shown (Phil. Mag. August 1887) that near the critical point in capillary tubes the relation of pressure to volume becomes fickle, the measurements of Andrews and Amagat differing from one another as much as experiment and calculation in Table IV. To illustrate this at higher volumes I introduced into Table III. a couple of Andrews's values of $Rvf(v)$ and $v^2\phi(v)$, from values of $Rf(v)$ and $\phi(v)$ calculated by Ramsay and Young (Phil. Mag. 1887), after conversion of Andrews's air-manometer indications to true metres of mercury. It will be seen that Andrews makes $\partial p/\partial T$ a little larger than Amagat; and this being so, it is not worth while to seek for closer agreement than that in Table IV., at least at present.

We have, however, a sensitive means of determining whether the form and the values of the constants adopted truly represent the behaviour of CO_2 closely enough at high volumes,—namely, Thomson and Joule's and Regnault's experiments on the cooling of CO_2 when it escapes through a porous plug from under pressure (Phil. Trans. 1854–1862; *Mém. de l'Acad.* xxxvii.). Natanson (Wied. *Ann.* xxxi.) has repeated the Joule and Thomson experiments on CO_2 under the more favourable conditions afforded by the commercial sale of the fluid in large quantity and great purity, so that he has been able to measure not only the cooling effect for a given pressure excess, but also its variation with pressure. Taking all these experiments together, we have a delicate test for the equation at high volumes.

The most convenient expression for the cooling effect for our present purpose is

$$K_p \frac{d\delta}{dp} = \theta \frac{dv}{d\theta} - v ;$$

where δ is the cooling effect, K_p is the specific heat at constant pressure, and θ is temperature on the absolute thermodynamic scale. In previous papers I took from Joule and Thomson's original investigation $\theta = T + \cdot 7^\circ$, not then aware that Sir W. Thomson, in his article "Heat" (*Encyc. Brit.*), had by a fuller discussion of all the experimental data proved $\theta = T$, and so removed the difficulty that the term $\cdot 7$ opposed to the harmony of the thermodynamic and molecular kinetic conceptions of temperature.

With our characteristic equation the cooling effect is, after the appropriate reductions, given by

$$K_p\frac{d\delta}{dp}=2(l/\mathrm{RT}-k)+p\{4(l/\mathrm{RT}-k)(l/\mathrm{RT}-2k)$$
$$-2k(l/\mathrm{RT}-2k)-lk/\mathrm{RT}\}/\mathrm{RT}.$$

Within Joule and Thomson's range of pressure this can be reduced to

$$K_p\frac{d\delta}{dp}=2(l/\mathrm{RT}-k)\ ;$$

and $d\delta$ can be made to stand for the integral cooling effect if dp stands for the integral excess of 2·54 metres of mercury, to which they reduced their results. The term in p will be taken account of when we come to Natanson's results. From Regnault's data we have, in dynamical measure

$$K_p=424(\cdot 187+\cdot 00027\,t).$$

TABLE V.

(Cooling effect of CO_2 escaping through a porous plug under a pressure excess of 2·54 metres of mercury.)

Temperature C.	7°·4.	8°.	19°·1.	35°·6.	54°.	91°·5.	93°·5.	97° 5.
Th. and Joule	4·4	4·2	3·9	3·4	2·95	2·35	2·16	2·13
Calculated	4·4	4·4	4·1	3·7	3·4	2·7	2·7	2·6
Temperature C.	−25°	3°.	100°.					
Regnault	6·3	4·1	2·6					
Calculated	5·5	4·5	2·6					

The agreement is as good as possible if both sets of experiments are taken into consideration. But Natanson's result affords a more delicate test; he found that at 20° up to 25 atmospheres the cooling effect for a pressure excess of one atmosphere could be represented by

$$\frac{d\delta}{dp}=1\cdot 18+\cdot 0126\,p\ ;$$

while the theoretical equation above gives

$$\frac{d\delta}{dp}=1\cdot 23+\cdot 012\,p,$$

which is practically identical with Natanson's. On account of the closeness of this agreement we obtain as an indirect conclusion, that the experimental work on CO_2 taken as a whole makes the absolute thermodynamic zero $-273°$, the same result as Sir W. Thomson has obtained for air and H in the article "Heat" (*Encyc. Brit.*), while for CO_2, using only Regnault's coefficient of expansion and Joule and his own cooling effects, he found $-273°\cdot9$. Now that CO_2 is seen to be in harmony with the other two more perfect gases, the number 273 may be accepted definitely as the absolute temperature of melting ice.

The equation therefore applies accurately at high volumes, a fact which we can prove by another test, seeing that Amagat carried out a special research (*Compt. Rend.* xciii.) to determine the ratios of pv to $p'v'$ at different temperatures and up to values of p' about 8 atmospheres, v being double v'.

TABLE VI.

(Values of $pv/p'v'$ at high volumes for CO_2.)

($p=5\cdot7$ metres of mercury : $v=2v'$.)

Temperature C. ……	50°.	100°.	200°.	300°.
Amagat………………	1·0145	1·0087	1·0040	1·0020
Calculated……………	1·0156	1·0092	1·0026	1·0000

As the experiments are not free from liability to an error of 1 in 1000, the agreement is again close enough to prove the applicability of the equation at high volumes.

In the sequel it will be shown that this equation applies to the great majority of compounds, but meanwhile the only other experimental determinations similar to those already discussed for ethyl oxide and carbonic dioxide are Amagat's for H_2, O_2, N_2, CH_4, and C_2H_4; Roth's for SO_2 and NH_3 (Wied. *Ann.* xi.); Janssen's for N_2O (Wied. *Beibl.* ii.); and Ramsay and Young's on methyl and ethyl alcohol (Phil. Mag. Aug. 1887). Our form of characteristic equation applies to SO_2, NH_3, and N_2O successfully, but not to H_2, O_2, N_2, and

CH_4, which require a still simpler type, the alcohols on the other hand requiring a less simple type. These are the values of k and l for SO_2, NH_3, and N_2O :—

	SO_2.	NH_3.	N_2O.
k . .	2·08	4·8	2·3
l . .	2740	22040	3420

with which the following pressures have been calculated for comparison with the experimental data, the latter being taken direct from air or nitrogen manometer without correction for departure from Boyle's law.

TABLE VII.

SO_2 at 99°·6.			NH_3 at 99°·6.			N_2O at 25°·1.		
v.	p exp.	p calc.	v.	p exp.	p calc.	v.	p exp.	p calc.
41·9	7·9	8·0	172	7·6	7·6	7·0	39·1	37·2
17·6	17·1	17·0	85·7	14·8	14·7	5·36	42·6	42·4
13·6	21·2	20·8	65·4	18·8	18·9	3·78	43·9	46·7
9·9	26·0	26·2	40·0	28·6	29·1			
At 183°.			At 183°.			At 43°·8.		
18·2	21·7	22·0	160	10·3	10·2	5·86	49·4	47·9
10·0	37·3	36·9	81·6	19·8	19·6	4·62	55·6	54·4
4·58	69·2	67·5	36·1	40·9	42·1	3·47	61·4	61·0
2·75	98·2	93·5	16·2	82·6	85·4	2·82	64·1	64·3

This comparison has been made only to show that the form is applicable to other bodies as well as to ethyl oxide and carbonic dioxide ; full confirmation of the form will come later on, in the study of many of its applications.

2. *Establishment of the Characteristic Equation for the Gaseous Elements,* with proof of continuity during liquefaction. —The simplest plan in the case of the gaseous elements will be to take nitrogen as typical and tabulate for it $Rvf(v)$ and $v^2\phi(v)$ from Amagat's experiments up to 320 metres of mercury.

TABLE VIII.—Nitrogen.

v.	$Rvf(v)$.	$v^2\phi(v)$.	v.	$Rvf(v)$.	$v^2\phi(v)$.
Perfect gas	2·233		4·15	3·25	1188
9·22	2·63		3·69	3·40	1007
6·91	2·77	1197	3·46	3·44	982
5·76	3·08	1396	3·23	3·80	1250
4·61	2·94	809			

The values of $v^2\phi(v)$ are unsteady, because the departures from Boyle's law are so small that $\phi(v)$ cannot be determined with accuracy; but it is clear enough that $v^2\phi(v)$ does not tend to diminish within the range of volume available, not a wide enough one, however, to convince us that there is a radical difference between the course of this function in elements and compounds. But if we adopt from this Table as it stands the only possible conclusion that $v^2\phi(v)$ is constant, we shall be able to justify it by its consequences. In contrast to the constancy of $v^2\phi(v)$ is the tendency of $Rvf(v)$ at low volumes to double its perfect gas-value.

In the case of H_2 and O_2 the two functions run a similar course to that for N_2, but it is a more unexpected fact that they also do the same for the compound methane, CH_4, as is shown in Table IX.

TABLE IX.—Methane.

v.	$Rvf(v)$.	$v^2\phi(v)$.	v.	$Rvf(v)$.	$v^2\phi(v)$.
Perfect gas.	3·908		12·11	5·24	6900
32·3	4·16		10·09	5·43	6400
28·2	4·30	5600	8·07	5·95	6500
24·2	4·39	6200	7·27	6·47	7000
20·2	4·73	6900	6·46	6·80	6800
16·1	4·73	6200	6·05	6·73	6200

It is evident that we have here to do with $v^2\phi(v)$ as a constant, that is with an internal virial varying inversely as the volume down to near the critical volume, and $Rvf(v)$ tending somewhere near that point to about double its value in the perfect gas state. The course of $Rvf(v)$ in these four gases is represented by the simple form

$$R\left\{1+\frac{k}{2}\Big/\left(v-\frac{k}{2}\right)\right\},$$

which attains the value $2R$ when $v=k$. Hence the characteristic equation down to $v=k$ is

$$pv=RT\left(1+\frac{\frac{k}{2}}{v-\frac{k}{2}}\right)-\frac{l}{v},$$

a form which I had already adopted for air (Phil. Mag. Aug. 1887). The following are the values for k and l :—

	H.	N_2.	O_2.	CH_4.	Air.
k . .	12·0	2·64	1·78	5·51	2·47[2·11]
l . .	41700	1175	851	6460	1110[910]

The values given in brackets for air are those previously found by me from Amagat's data (*Compt. Rend.* xcix.), but as these data are not carried to such high pressures as those for N_2 and O_2, I have calculated values for air by adding to four fifths of the values for N_2 one fifth of the values for O_2.

This equation is almost identical with that of Van der Waals, but it is a little simpler. It gives the following pressures for comparison with Amagat's experimental results :—

TABLE X.

Hydrogen. At 17·7° C.			Nitrogen. At 17·7° C.			Oxygen. At 14·7° C.		
v.	p exp.	p calc.	v.	p exp.	p calc.	v.	p exp.	p calc.
166·9	57·5	56·9	13·83	46	45·7	5·73	88·9	89·7
100·1	99	99	6·91	92	91	3·58	141	142
60·1	176	176	4·61	145	142	2·58	201	203
46·7	238	241	3·69	194	188	2·43	216	219
			3·23	223	226			
At 100° C.			At 100° C.			At 100° C.		
166·9	74·2	73·3	13·83	60	60·5	5·73	123	124
100·1	129	128	6·91	125	124	3·58	204	204
60·1	230	229	4·61	200	199	2·58	300	301
46·7	311	315	3·69	270	266	2·43	322	327
			3·23	320	323			

The experimental numbers for oxygen are taken from Amagat's data in the *Comptes Rendus*, xci.

The most delicate test we can apply to our form at high volumes is, in the case of air, to compare the calculated with the experimental Thomson and Joule cooling effect. When I did this with the previous equation for air ($k=2{\cdot}11$ and $l=910$), I assumed the difference ·7° to exist between the melting-point of ice on the thermodynamic and gas thermometers; but, as already pointed out, Sir W. Thomson having proved this difference not to exist, there must have been a compensation of errors in the application of the previous equation. Thomson's expression for the cooling effect, applied to our equation for air, becomes

$$K_p d\delta/dp = 2l/RT - k/2,$$

which gives the following calculated values :—

Cooling effects of air escaping through a porous plug into the atmosphere under a pressure excess of 2·54 metres of mercury.

Temperature C.	7°·1.	17°.	39°·5.	92°·8.
Experiment	·88	·86	·75	·51
Calculation	·84	·80	·71	·55

The agreement is the closest to be looked for, and proves the accuracy of our equation for air at high volumes.

At low volumes we can test the form for all the elementary gases and CH_4 by applying it to the calculation of the critical volume, pressure, and temperature in each case. To do this at the present stage we must assume that our form can be trusted to hold not only to the critical volume but also a little past into the liquid region, a legitimate assumption for the elements, where we have seen the internal virial varying inversely as the volume, and so giving a guarantee of continuity, but not legitimate for the compounds where discontinuity occurs. Then, applying James Thomson's idea of the passage from the gaseous to the liquid state, as precisionized by Maxwell and Clausius, we have the critical point determined by the conditions $\partial p/\partial v=0$, $\partial^2 p/\partial v^2=0$.

Along with the characteristic equation these lead to the following values:—critical volume $v_c=3k/2$; critical temperature $T_c=16l/27Rk$; critical pressure $p_c=4l/27k^2$,—to compare with the experimental values found by Olszewski for O_2 and N_2 (*Compt. Rend.* c.), by Wroblewski for air, and by Dewar for CH_4 (Phil. Mag. 1884, xviii.).

TABLE XI.

		H_2.	N_2.	O_2.	CH_4.	Air.
Critical Volume.	exper. ...	......	3·4	2·5		
	calc.	......	3·96	2·67		
Critical Temperature.	exper. ...	......	−146	−119	−99·5	−140
	calc.	−229	−155	−127	−95	−149
Critical Pressure.	exper. ...	......	27	38	37	30
	calc.	19	25	40	32	27

The agreement is all that can be looked for in view of the difficulties of measuring these low critical temperatures and their associated pressures. With regard to hydrogen all we know is that Olszewski (*Compt. Rend.* ci.) has submitted it to a temperature estimated by him as −220° without a sign of liquefaction. If our equation is to be trusted, it would indicate that he would need to go some 10 degrees lower before the only unliquefied gas is conquered. Wroblewski has published data on the compressibility of H_2 up to pressures of 70 atmospheres at temperatures of −103° and −182° (Journ. Chem. Soc. 1889), and with these our equation is in accord, but there is hardly need of tabulated proof.

3. *Brief discussion of exceptional Compounds such as the Alcohols and Ethylene.*—To complete our survey of the experimental material on bodies above the critical region we have to consider Ramsay and Young's observations on methyl and ethyl alcohol, and Amagat's on ethylene. Ramsay and Young point out that at low volumes the values of $\partial p/\partial T$ for the alcohols are not so reliable as for ethyl oxide, being determined from a smaller temperature range; hence our values of $Rvf(v)$ and $v^2\phi(v)$ are not so reliable as before, but they suffice to show the exceptional nature of these bodies.

TABLE XII.—Methyl Alcohol.

v.	$\mathrm{R}vf(v)$.	$v^2\phi(v)$.	v.	$\mathrm{R}vf(v)$.	$v^2\phi(v)$.
Perfect gas.	1·950		25	3·50	24800
340	2·14		20	3·70	23000
240	2·32	45500	18	3·79	22000
200	2·40	48000	16	3·97	21700
170	2·46	46100	14	4·18	20900
135	2·56	43700	12	4·38	19900
100	2·70	41000	11	4·46	19000
70	3·04	42100	10	4·53	17900
50	3·10	33500	9	4·58	16700
40	3·20	30200	8	4·55	15000
30	3·37	26800	7	4·39	12900

Ethyl Alcohol.

v.	$\mathrm{R}vf(v)$.	$v^2\phi(v)$.	v.	$\mathrm{R}vf(v)$.	$v^2\phi(v)$.
Perfect gas.	1·35		8	3·43	11300
108·2	1·65	18000	6	3·34	8600
53·4	1·83	15000	4	3·37	6200
30	2·16	15000	3	3·51	5000
26·1	2·31	16000	2·4	3·93	4600
18·2	2·83	17000	2	4·87	4700
12	3·32	15000	1·8	5·83	4900
10	3·41	13500	1·6	7·06	4900
9	3·43	12500	1·4	8·81	4700

The numbers for methyl alcohol do not extend as far as the critical volume, while those for ethyl alcohol go considerably beyond it, lying as it does between 3 and 4; but we notice in both cases that $\mathrm{R}vf(v)$ increases from the limiting gaseous value, but attains a practically constant value before the critical volume is reached. In ethyl alcohol we may say that the value 3·4 is retained from volume 10 to volume 3, and moreover this 3·4 is not now double the initial 1·35, but about 2·5 times it. In methyl alcohol the value 4·5 may be said to be retained constant from volume 11 to 7, the lowest on the table; so that it is probable that, as in the case of ethyl alcohol, this value will be retained down to the critical volume: here again, also, the 4·5 is more than double the initial 1·95, but is only 2·3, not 2·5, times it.

Note that, in ethyl alcohol, as soon as the critical volume is passed $Rvf(v)$ begins again to increase rapidly, just as happened in the case of ethyl oxide. But for our present purpose more interest attaches to the course of $v^2\phi(v)$. In methyl alcohol at high volumes it seems to approach a limit which we may assign as 46,000, and then at volume 16, where $Rvf(v)$ has risen to double its initial value, $v^2\phi(v)$ has fallen to almost half of 46,000, but as $Rvf(v)$ continues to rise $v^2\phi(v)$ continues to fall, and still continues to do so even when $Rvf(v)$ has become constant. In methyl alcohol we cannot follow the changes right down to the critical volume, but in ethyl alcohol we see that $v^2\phi(v)$ attains at the critical volume a value which is carried constant into the liquid state, this constant value being about one quarter of the apparent limiting value 20,000 at large volumes. The constancy of $v^2\phi(v)$ below the critical volume is in striking contrast to the rapid variation of $Rvf(v)$.

I have not sought to represent by formulas the course of the two functions for the alcohols, as I have doubts about $\partial p/\partial T$ being independent of temperature in the case of the alcohols; if it is variable then the values of our functions are affected with error. In any case we have seen that above the critical region the alcohols behave differently from our two typical compounds, ethyl oxide and carbonic dioxide; in section 14 it will be seen that in the liquid region, on the other hand, the alcohols approach the regular compounds in many respects, but are still exceptional in others. There remains now only ethylene to consider as to its gaseous behaviour.

TABLE XIII.—Ethylene.

v.	$Rvf(v)$.	$v^2\phi(v)$.	v.	$Rvf(v)$.	$v^2\phi(v)$.
Perfect gas.	2·22		4·61	4·15	4500
20·75	2·71	5800	4·15	4·45	4500
16·14	2·78	5320	3·69	4·81	4400
11·53	3·07	5500	3·22	5·71	4700
9·23	3·24	5300	2·77	6·41	4400
6·92	3·58	5100	2·65	6·63	4200
5·76	3·83	4900	2·54	7·87	5000

According to Cailletet and Mathias (*Compt. Rend.* cii.), the critical volume of ethylene is about 4·5; so that, again, in the above table we see $Rvf(v)$ near the critical volume attaining double its initial value and increasing rapidly thereafter. Once more, too, we see $v^2\phi(v)$ attaining near the critical volume a value which it retains constant below; but ethylene is exceptional in that this value is not half the limit at high volumes. The facts in the above table may be summarized in the statements that $Rvf(v)$ may be represented by the form $R(1+k/v)$, and $v^2\phi(v)$ by the form $vl/(v+a)$; so that the characteristic equation for ethylene is

$$pv = RT\left(1+\frac{k}{v}\right)-\frac{l}{v+a},$$

with $k=4{\cdot}15$, $a=1{\cdot}64$, and $l=6270$.

The form for ethylene is intermediate in simplicity between that for the simple gases and that for compounds, except that it has an extra constant. It is also worth noting that the forms

$$\frac{k}{2}\Big/\left(v-\frac{k}{2}\right),\quad k/v,\quad \text{and}\quad 2k/(v+k)$$

are special cases of a general form

$$nk/\{v+(n-1)k\},$$

with $n=\frac{1}{2}$, 1, and 2.

4. *Establishment of Characteristic Equation below the region of the Critical Volume.*—Now that we have practically exhausted the available data of the gaseous state, we see that by themselves they do not give much scope for generalization; but if we can secure an equation applicable from the critical volume down to the volumes of liquids in the ordinary state, then, with two equations covering almost the whole range of fluidity, we shall have a much larger experimental area laid under contribution for information on the characters of molecules.

Already we have secured one important fact towards the acquisition of such an equation, namely that below the critical volume the internal virial term varies inversely as the volume; and in the case of ethyl oxide we know its actual amount $l/2v$ with $l=5514$. We have therefore only to add to $l/2v$ Ramsay and Young's values of pv at different temperatures for different volumes below the critical, and we obtain the values of the kinetic-energy term in the desired equation ; we can then proceed to study how this quantity depends on temperature and volume, and express the resulting conclusions in a formula.

As to the form we have this clue, that it must join on continuously with the previous one where that ceases to be applicable. Now the first fact to notice is that our form for compounds above the critical region cannot, like that for the elements, give a critical point by itself at all ; for given p and T it is not a cubic but a quadratic in v, and hence cannot give us the three equal roots which are adopted as characteristic of the critical point when we apply the conditions

$$\partial p/\partial v = 0, \quad \partial^2 p/\partial v^2 = 0.$$

This emphasizes the discontinuity in compounds as contrasted with elements. However, we know as an experimental fact that at the critical point $\partial p/\partial v=0$, which with the characteristic equation gives us only two relations between the critical temperature, pressure, and volume. As a third relation that would perfectly define these three quantities I was led to believe that the critical volume is proportional to k, and found that

$$\text{critical volume } v_c = 7k/6$$

is the relation which, with the two others, gives successfully the numerics of the critical state in agreement with experiment. As this will be proved subsequently (Section 10) for a large number of substances, I will not delay at present to give examples, except for those compounds for which we have already found k and l.

TABLE XIV.

Critical temperature, $T_c = 120\,l/409\,Rk$; critical pressure, $p = 36\,l/409\,k^2$.

		$(C_2H_5)_2O$.	CO_2.	SO_2.	NH_3.	N_2O.
Critical Temperature.	exper. ...	194	32	155	130	35
	calc.......	199	52	125	96	36
Critical Pressure.	exper. ...	27·1	59	60	87	57
	calc.......	29·3	78·6	56	84	57

The want of accuracy in the agreement in parts of this table is to be ascribed partly to inaccuracy in the ordinary determinations of the critical point, as I have already pointed out that capillary action must sometimes largely affect the numerics of the critical state when these are determined in capillary tubes (Phil. Mag. August 1887). Regnault, in his account of his experiments on the saturation-pressures of CO_2, expressly declares that he had liquid CO_2 at 42°, which is 10° above the apparent critical temperature in capillary tubes; and Cailletet and Colardeau (*Compt. Rend.* cviii.) have shown that although the meniscus between gas and liquid CO_2 disappears to the eye about 31° or 32°, yet characteristic differences between liquid and gas can be proved to exist several degrees higher than this. Hence an error of at least 10° is possible in ordinary determinations of critical temperatures. On the other hand, an error of 5 per cent. in the value of an absolute temperature of about 400° as given by our equation would amount to 20°. Table XIV. is to be taken in the light of these facts.

We have now ascertained a second property that our equation for volumes below the critical is to possess: it must begin to apply when $v=7k/6$, as the other form cannot apply below this volume at the critical temperature. At this volume the kinetic-energy term in our form above the critical region becomes

$$\mathrm{RT}(1+12/13), \quad \text{or} \quad 25\mathrm{RT}/13;$$

so that $25\mathrm{R}/13$ is the lower limit of the term which in the new equation is to take the same place as $\mathrm{R}vf(v)$ hitherto. Hence for this term the form

$$25\mathrm{R}(1+\mathrm{F}(v))/13$$

naturally suggests itself, and as $\mathrm{F}(v)$ is to vanish when $v=7k/6$, we get $(7k/6-v)/\psi(v)$ as a suggestion for its form; and it only remains from the data obtained, as I have said, by adding $l/2v$ to Ramsay and Young's values of pv for volumes of ethyl oxide below k, to determine the form of the function $\psi(v)$. This was found, after a rather tedious search, to come out in the simple form

$$\sqrt{\mathrm{T}}(v-\beta)\ \mathrm{B};$$

so that finally we have the following as the equation for ethyl oxide below the volume k:

$$pv = \mathrm{R'T}\left(1 + \frac{\sqrt{\mathrm{T}}}{\mathrm{B}} \cdot \frac{k'-v}{v-\beta}\right) - \frac{l}{2v},$$

with the following values for the constants:

$$\mathrm{R'} = 25\mathrm{R}/13, \quad k' = 7k/6, \quad \mathrm{B} = 63{\cdot}1, \quad \beta = 1{\cdot}11,$$

R, k, and l as before.

I propose to call this the infracritical equation. It is to be noticed that we have introduced only two additional constants; so that, as regards number of constants, we could hardly look for a simpler form.

Above the volume $7k/6$ the appropriate form was proved to be

$$pv = \mathrm{RT}\left(1 + \frac{2k}{v+k}\right) - \frac{l}{v+k},$$

which I propose to call the supracritical equation.

Between k and $7k/6$ we have the circacritical form

$$pv = \mathrm{R'T}\left(1 + \frac{\sqrt{\mathrm{T}}}{\mathrm{B}} \cdot \frac{k'-v}{v-\beta}\right) - \frac{l}{v+k}.$$

This, then, gives the complete representation of ethyl oxide in the fluid state if we establish the sufficiency of the infracritical form, as we now proceed to do. In the next table are compared the pressures found by Ramsay and Young and those given by the equation.

TABLE XV.—Liquid Ethyl Oxide.

Volume		3·7.	2·75.	2·25.	2.	1·9.
195° C.	Pressure, exper. .	28	29	43		
	„ calc. ...	32	26	45		
175° C.	Pressure, exper. .	......	......	19	43	
	„ calc. ...	......	......	14	48	
150° C.	Pressure, exper. .	......	......	......	......	19·5
	„ calc. ...	......	......	......	......	20

For the proper appreciation of this table it must be borne in mind that as soon as we enter the liquid region the pv term of the characteristic equation becomes the small difference of two terms, a small percentage error in either of which becomes a large one in *pv*. The fact that the above table brings out is that from 150° to 195° the relation between volume and temperature given by the equation is so accurate as to make only the small errors in pressure in the above table. But to show this more directly, we will now compare the volumes of the liquid under a pressure of 9 metres of mercury between 0° and 100°, as determined by Grimaldi (Wied. *Beibl.* x.) and as given by the equation. The specific gravity of ethyl oxide at 0° and under one atmosphere is taken as ·7366.

TABLE XVI. ($p = 19{\cdot}5$ metre.)

Temperature	0°.	50°.	100°.	150°.
Volume, experiment.........	1·355	1·469	1·630	1·9
„ calculated	1·354	1·467	1·633	1·9

This, taken in conjunction with Table XV., shows that the equation represents with a high degree of accuracy the expansion of liquid ethyl oxide right up to the critical volume. It is now to be tested as to its power to give compressibilities correctly. The next Table contains the calculated compressibilities of liquid ethyl oxide, and also the experimental as given by Amagat (*Ann. de Chim. et de Phys.* 5 sér. t. xi.), Avenarius (Wied. *Beibl.* ii.), and Grimaldi (Wied. *Beibl.* x.). Amagat's values had to be interpolated for comparison with the others.

TABLE XVII.

Compressibilities with metre of mercury as pressure-unit.

Temperature	0°.	40°.	60°.	100°.
Amagat	·000200	·000309	·000380	·000730
Avenarius	·000178	·000317	·000403	·000654
Grimaldi	·000207	·000316	·000407	·000632
Equation	·000183	·000300	·000392	·000710

The agreement here is again satisfactory, and we have now seen that our form, with only two constants in addition to those characteristic of the gaseous state, can give both the expansion and compression of the liquid at low pressures; but Amagat has measured these also at high pressures up to 2000 and 3000 atmos. (*Compt. Rend.* ciii. and cv.), and the following Table compares first his values of the mean coefficient of expansion between 0° and 50° at pressures from 76 up to 2280 metres with those given by the equation, and, second, his values of the mean compressibility at 17°·4 and at pressures up to 1500 metres with those given by the equation. If v_1 and v_2 are the volumes at p_1 and p_2, then the mean compressibility is taken as $(v_1-v_2)/v_1(p_2-p_1)$. The apparent compressibilities given by Amagat are converted to true values by adding ·000002, which he has since given as the compressibility of glass.

TABLE XVIII.

Mean Coefficient of Expansion at high pressures.

p in metres ...	76.	380.	760.	1140.	1520.	1900.	2280.
Amagat	·00170	·00112	·00091	·00077	·00070	·00063	·00056
Equation	·00170	·00101	·00076	·00066	·00056	·00050	·00047

Mean Compressibilities at high pressures.

p in metres ...	76 to 114	114 to 366	366 to 654	654 to 933	933 to 1218	1218 to 1500
Amagat	·000208	·000143	·000112	·000086	·000070	·000062
Equation	·000197	·000128	·000085	·000060	·000046	·000037

As regards expansion the equation goes fairly near to the truth; except at the lowest pressures, it gives coefficients somewhat smaller than the experimental, but it parallels closely the main phenomenon of the rapid diminution of the coefficient with rising pressure. But in the compressibilities there is an increasing divergence between experiment and equation with increasing pressure, although again the equation is true to the main fact of the rapid diminution of compressibility with increasing pressure. We may conclude from the last table that our equation holds within the limits of experimental accuracy up to 760 metres; beyond that it begins to fail. A simple empirical modification would adapt the form to the whole of Amagat's range, but as it stands it will be found good enough for our applications.

We will now consider briefly how this form applies to carbonic dioxide below the critical volume; and the comparison is interesting, as it relates to temperatures both above and below the critical. The values of the constants are $B=54$, $\beta=\cdot692$.

TABLE XIX.

Carbonic Dioxide below critical volume.

Volume		1·526.	1·203.	1·115.	1·027.
70° C.	Pressure, exp......	150	274		
	„ calc. ...	152	266		
35° C.	Pressure, exp.......	69	126	187	320
	„ calc. ...	69	120	184	320
18° C.	Pressure, exp.......	......	......	99	200
	„ calc. ...	......	......	97	208

The agreement is within the limit of experimental error at the high pressures. Cailletet and Mathias have determined (*Compt. Rend.* cii.) the density of liquid CO_2 at various temperatures under the pressure of saturation. Here is a comparison with a couple of their results:—

Temperature . . .	−34°.	0 .
Volume—Cailletet and Mathias .	·946	1·087
„ Equation	·943	1·086

As far as compound gases are concerned, the applicability of the form for volumes below the critical has now been demonstrated in two typical cases. The elementary gases have now to be considered as to their behaviour below the critical volume. The data are again those furnished by Amagat (*Compt. Rend.* cvii. and Phil. Mag. Dec. 1888) on the compressibility of these gases between 760 and 2280 metres of mercury. Our study of these bodies above the critical volume has given us the knowledge that the internal virial term below k must be l/v, and the kinetic-energy term at the critical volume is 3RT 2, and with these guides the complete form required is soon found from the experimental numbers. It is

$$pv = \tfrac{3}{2}\mathrm{RT}\left(1+b\,\frac{\frac{3}{2}k-v}{v-\beta}\right)-\frac{l}{v},$$

with the following values for the additional constants β and b :—

	k.	β.	b.
Hydrogen . . .	12	4·3	·480
Nitrogen . . .	2·64	·81	·420
Oxygen . . .	1·78	·604	·4415
Methane . . .	[5·51]	[1·59]	[·447]

The approximate equality of the values of the constant b is worth noting. I have also reproduced here the values of k at the side of those for β, in order to point out that β is nearly $k/3$ in each case. Amagat has not published data for methane at volumes below the critical region, but the numbers given in brackets for methane were obtained indirectly as explained below. These relations of β and b give our equation such a degree of simplicity as largely to establish the soundness of its form. The next Table shows the degree of accuracy with which it represents the experimental facts.

TABLE XX.—Oxygen at high pressures.

	Volume	1·277.	1·097.	1·008.	·949.	·905.
15° C.	Pressure, exp.......	760	1140	1520	1900	2280
	„ calc. ...	739	1141	1512	1893	2296

The agreement is quite as good for hydrogen and nitrogen.

By means of this equation we can calculate the volumes of a gramme of liquid nitrogen and oxygen at their boiling-points under a pressure of ·76 metre, for comparison with Wroblewski and Olszewski's determinations of the same (*Compt. Rend.* cii.; Wied. *Beibl.* x.; and 'Nature,' April 1887).

		Oxygen.		Nitrogen.
Volume at −184°	Wroblewski . .	·85	at −194°	1·20
	Olszewski . .	·89		1·13
	Equation . .	·90		1·26

The equation is seen to give the volumes of these two bodies at these low temperatures within the present limits of experimental accuracy, and accordingly it covers a range of 2000-metres pressure and almost the whole experimental range of temperature. In the case of methane, if we take Olszewski's value 2·41 for its volume at −164°, and assume b is the mean of b for H_2, N_2, and O_2, then we can calculate the value of β which is tabulated above.

To ethylene above the critical region we had to assign a special form intermediate between that for ordinary compounds and that for elements; so that we had better do likewise for its infracritical equation, which I have cast in the form

$$pv=(1+\tfrac{1}{8})\,\mathrm{RT}\left(1+\frac{\sqrt{\mathrm{T}}}{\mathrm{B}}\cdot\frac{\frac{5}{4}k-v}{v-\beta}\right)-\frac{l}{v}\cdot\frac{k}{k+a},$$

with $\mathrm{B}=56{\cdot}5$, $\beta=1{\cdot}53$. For the elements we had $v_c=3k/2$, for ordinary compounds $v_c=7k/6$; so that to make ethylene intermediate v_c is taken as $5k/4$, all these being of the general form $(1+2n)/2n$, with $n=1, 2, 3$.

With the values of the constants B and β given above as derived from Amagat's results at high pressures, we can determine the density of liquid ethylene; at −21° under saturation-pressure the density is ·414, identical with the experimental value of Cailletet and Mathias (*Compt. Rend.* cii.).

It will be as well at this stage to extract clear from among the argumentative detail the most important results so far obtained.

First, in the elements the internal virial varies inversely as the volume over the whole experimental range.

Second, in compounds there is mathematical discontinuity in the value of the internal virial at volume k; from volume k downwards the internal virial varies inversely as the volume: from the volume k upwards it tends towards variation inversely as the volume as the limiting law, the limiting constant being double that which holds below the volume k; between the two limiting cases the internal virial of compounds varies inversely as $(v+k)$.

Third, a fact of the highest importance in connexion with he kinetic-energy or temperature term in the equation arrests our attention, namely, that the coefficient of T in it, or the apparent rate of variation of the translatory kinetic energy with temperature at constant volume, attains near the critical volume double its value in the gaseous state, and below the critical region increases rapidly with diminishing volume (see column $Rvf(v)$ in Table I.), becoming at ordinary liquid volumes as much as ten times as large (see coefficient of T in infracritical equation). Now the specific heat of liquids at constant volume, which is the rate of variation of the total energy with temperature, is rarely much more than twice that for their vapours. Hence we must seriously consider the interpretation to be put on the different terms of our equations.

5. *A short digression on the general interpretation of Clausius's Equation of the Virial.*—Returning to Clausius's theorem of the virial,

$$\tfrac{3}{2}pv = \Sigma\,\tfrac{1}{2}mV^2 - \tfrac{1}{2}\,.\,\tfrac{1}{2}\Sigma\Sigma\,Rr,$$

we see that strictly the kinetic-energy term includes not only the energy of the motion of the molecules as wholes, but also that of the motion of their parts, and at the same time the internal virial includes the actions between the parts of the molecules as well as those between the molecules. Calling these actions the chemic force, we can write the theorem thus:

$$\tfrac{3}{2}pv = \text{the total kinetic energy} - \text{chemic virial} - \text{virial of molecular forces.}$$

Now in the usual treatment of the equation it is assumed that the chemic virial is equal to that part of the total kinetic energy which is due to the motion of the parts of the molecules relatively to their centres of mass, and neutralizes it in our equation, reducing it to

$$\tfrac{3}{2}pv = \text{translatory kinetic energy of molecules as wholes} - \text{virial of molecular forces.}$$

But if we retain the full equation, and assume that the virial term we have been finding for various bodies is the true virial of the molecular forces, and includes none of the chemic virial, then the term usually regarded as the translatory kinetic energy of the molecules as wholes is really the total kinetic energy minus the chemic virial.

Let E be the total kinetic energy of unit mass, V the virial of the chemic forces, and P their potential energy ; then, above the critical volume,

$$E-V=\tfrac{3}{2}RT\left(1+\frac{2k}{v+k}\right)$$

and

$$\frac{\partial}{\partial T}(E-V)=\tfrac{3}{2}R\left(1+\frac{2k}{v+k}\right),$$

which in the limiting gaseous state becomes 3R/2.
Also

$$\frac{\partial}{\partial T}(E-P) = K_v,$$

the specific heat at constant volume.

Below the critical volume,

$$E-V=\tfrac{3}{2}R'T\left(1+\frac{\sqrt{T}}{B}\cdot\frac{k'-v}{v-\beta}\right),$$

$$\frac{\partial}{\partial T}(E-V)=\tfrac{3}{2}R'+\tfrac{9}{4}R'\frac{\sqrt{T}}{B}\cdot\frac{k'-v}{v-\beta};$$

and, again,

$$\frac{\partial}{\partial T}(E-P) = K_v.$$

Now we can calculate K_v from the experimental values of K_p by the relation

$$K_v=K_p-T\frac{\partial p}{\partial T}\frac{\partial v}{\partial T}.$$

Let us then make a comparison in the case of ethyl oxide, using E. Wiedemann's value ·3725 for K_p for the vapour at 0°, and Regnault's value ·529 for the liquid at 0° ; then, converting to ergs per degree C., we get

	Vapour at 0°.	Liquid at 0°.
$\frac{\partial}{\partial T}(E-V) =$	$1{\cdot}68\times10^6$	$15{\cdot}3\times10^6$,
K_v or $\frac{\partial}{\partial T}(E-P) =$	$14{\cdot}4\times10^6$	$17{\cdot}9\times10^6$.

Thus we see that while in the liquid $\partial(E-V)/\partial T$ is nearly

equal to $\partial(E-P)/\partial T$, there is a great difference in the vapour:
$\partial(V-P)/\partial T=12{\cdot}7$ for the vapour and only $2{\cdot}6$ for the liquid. Or, while K_v is nearly the same in the two states, $\partial(E-V)/\partial T$ has in the liquid state increased to nine times its value in the vaporous. We have here, therefore, an interesting opening into the regions of chemic force; but meanwhile we must restrict ourselves to the question of molecular force at present in hand, calling attention, however, to the fact that our energy term in its two forms for elements and its two forms for compounds is well worthy of the closest study. It summarizes a lot of information about the internal dynamics of molecules—perhaps about the relations of matter and æther; but these would need to be extracted by a special research on the term and its relation to our experimental knowledge of specific heat. It is worth mentioning here that Clausius's equation of the virial, as usually applied to molecular physics, takes no account of the mutual action of matter and æther—an action which we know must exist, from the radiation of heat by gases as well as by liquids and solids. According to ordinary views of the æther this may be neglected, on account of the smallness of the mass and of the specific heat of the æther; but it is well to remember that we are neglecting it.

6. *Consideration of Van der Waals's generalization.*—We are now in a position to consider how far Van der Waals's generalization holds, namely:—If the volume, pressure, and temperature are measured for each substance in terms of the critical values as units, then one and the same law holds for all substances.

In the first place, we see from what has gone before that the same law cannot apply to both elements and compounds, nor can the alcohols and water follow the same law as regular compounds.

In the case of the elements and methane we have the critical volume, pressure, and temperature given in terms of R, l, and k by three relations (see end of Section 2),

$$v_c=3k/2, \quad p_c=\frac{4}{27}\frac{l}{k^2}, \quad T_c=\frac{16}{27}\frac{l}{Rk}.$$

Whence, in the supracritical equation replacing R, l, and k by their values in terms of v_c, p_c, T_c, we get

$$\frac{pv}{p_c v_c}=\frac{8}{3}\,\frac{\mathrm{T}}{\mathrm{T}_c}\left(1+\frac{1}{\frac{3v}{v_c}-1}\right)-\frac{3v_c}{v};$$

which shows that when the critical values are made the units in the measurements of the variables, one and the same law holds for the elements above the critical volume.

Below the critical volume we have

$$\frac{pv}{p_c v_c}=\frac{9}{16}\,\frac{\mathrm{T}}{\mathrm{T}_c}\left(1+b\,\frac{1-\frac{v}{v_c}}{\frac{v}{v_c}-\frac{\beta}{v_c}}\right)-\frac{3_c}{v}.$$

We have seen that b is nearly the same for these bodies and that β/v_c is approximately constant, so that below the critical volume the elements and methane all follow approximately the same law.

In the case of compounds, we have (see Section 4, at the beginning)

$$v_c=7k/6,\quad p_c=\frac{36}{409}\,\frac{l}{k^2},\quad \mathrm{T}_c=\frac{120}{409}\,\frac{l}{\mathrm{R}k},$$

with which, eliminating R, k, and l from the supracritical equation, we get

$$\frac{pv}{p_c v_c}=\frac{20}{7}\,\frac{\mathrm{T}}{\mathrm{T}_c}\left(1+\frac{2}{\frac{7}{6}\frac{v}{v_c}+1}\right)-\frac{409}{42}\,\frac{1}{\frac{7}{6}\frac{v}{v_c}+1}.$$

Hence, above the critical volume the compounds follow the same law among themselves.

In the same way, below the critical volume we get for compounds:—

$$\frac{pv}{p_c v_c}=\frac{20}{7}\cdot\frac{25}{13}\,\frac{\mathrm{T}}{\mathrm{T}_c}\left(1+\frac{\sqrt{\mathrm{T}}}{\mathrm{B}}\cdot\frac{1-\frac{v}{v_c}}{\frac{v}{v_c}-\frac{\beta}{v_c}}\right)-\frac{409}{98}\,\frac{v_c}{v}.$$

One and the same law holds for compounds below the critical volume only if B varies as the square root of the critical temperature, and if β is proportional to the critical volume : in the elements we have found the latter condition to hold approximately, and so we are prepared to find it do so for compounds. The following Table compares B with $\sqrt{T_c}$ and β with k, which is $6v_c/7$ for the five compounds for which we have as yet found k. The values of B and β for NH_3 and N_2O were obtained from Andréeff's data for the expansion of these bodies as liquids (*Ann. Chem. Pharm.* cx.) and for SO_2 from Jouk's (Wied. *Beibl.* vi.).

Table XXI.

	k.	β.	k/β.	B.	$B/\sqrt{T_c}$.
$(C_2H_5)_2O$	4·066	1·11	3·7	63	2·9
CO_2	1·76	·69	4·0	54	3·0
SO_2	2·08	·55	3·8	71	3·6
NH_3	4·8	1·22	3·9	70	3·6
N_2O	2·3	·66	3·5	55	3·1

In these bodies we find a fair approximation to proportionality between β and k on the one hand, and between B and $\sqrt{T_c}$ on the other ; to the same degree of approximation Van der Waals's generalization can be applied to compounds below the critical volume (excluding of course such exceptional bodies as the alcohols and water).

The accurate statement of the generalization ought then to be as follows :—When the variables are expressed in terms of their critical values as units, then down to the critical point compound bodies with certain exceptions have all one and the same characteristic equation, but below the critical point they have closely similar but not identical equations.

It is a remarkable fact that Van der Waals should have been led to his valuable generalization by means of a form of equation which completely fails to apply to the substances which are the subject of the generalization. As a point in the history of this branch of molecular physics, it calls for mention that Waterston, in the Phil. Mag. vol. xxxv. (1868), had practically discovered the generalization, and expressed it in its most striking aspects by means of several diagrams for a number of bodies; but the verbal expression of his results was so unsystematic, and withal so crabbed, that his work has been overlooked.

There is one typical application of the generalization which is of special importance—to the relation between pressure and temperature of saturation. If with the aid of our equations we trace the complete isothermals for temperatures below the critical, we shall get curves with the James Thomson double-bend as shown in Ramsay and Young's isothermals for ether (Phil. Mag. May 1887).

According to Maxwell's thermodynamical deduction, the pressure of saturation at a given temperature is that corresponding to the line of constant pressure which cuts off equal areas in the two bends, a result which Ramsay and Young verified by actual measurement on their curves.

Let v_3 and v_1 be the volumes of saturated vapour and liquid at pressure P and temperature T; then Maxwell's principle gives us that the pressure of saturation is defined by the three equations:—

$$\mathrm{P}(v_3-v_1)=\int_{v_1}^{v} p\,dv,$$

$$\mathrm{P}v_3=\mathrm{RT}\left(1+\frac{2k}{v_3+k}\right)-\frac{l}{v_3+k},$$

$$\mathrm{P}v_1=\tfrac{25}{13}\,\mathrm{RT}\left(1+\frac{\sqrt{\mathrm{T}}}{\mathrm{B}}\,\frac{\frac{7}{6}k-v_1}{v_1-\beta}\right)-\frac{l}{2v_1}.$$

v_3 is given in terms of P and T by a quadratic, and can therefore be eliminated from the integral when evaluated; v_1 is given by a cubic, but as $\mathrm{P}v_1$ can for practical purposes be put 0, a very close approximation to v_1 can be obtained also from a quadratic. The resulting relation between P and T, which is the law of saturation, involves the constants R, k, l, B, and β.

The actual evaluation of the integral would of course proceed in three stages, corresponding to the supra-, circa-, and infracritical equations. The law of the integral in the first stage from v_3 to $7k/6$, with critical values of the variables as units, would be the same for all compounds; and we have seen that the integral in the other two stages will follow approximately the same law in all cases. Hence if saturation pressures and temperatures are expressed in terms of the critical values, the law of their relation will be approximately the same for all regular compounds.

If this were an absolutely accurate relation, the best means of testing it would be to take Regnault's formula, with one exponential term,

$$\log p = a + b\alpha^t,$$

and by a simple recalculation from his values for a, b, and α, to cast it in the form

$$\log p/p_c = e + d\gamma^{T/T_c},$$

proving that the constants e, d, and γ are approximately the same for all compounds. But the objection to this plan soon becomes obvious on trial, as the formula owes its empirical convenience to the power of adjustment amongst the constants; and the same difficulty would be experienced with any purely empirical equation.

Accordingly, to test this matter, I have thought it best to compare the pressures of a number of bodies at temperatures which are constant fractions of their critical temperatures, such as $\cdot 6\,T_c$, $\cdot 7\,T_c$, and so on. The ratio of the pressure of any substance to the corresponding pressure of ethyl oxide ought to be approximately the same for that substance at all values of the fraction. Great uncertainty attends the measurement of critical pressures: an error of 20° in the critical temperature is not a large fraction of its value measured from absolute zero, but it makes a large difference in a saturation-pressure, and the critical pressure is the limiting saturation-pressure. In the subjoined Table the critical-pressure ratios are given for what they are worth in the column T_c.

This table makes it clear enough that, in applying Van der Waals's generalization below the critical volume, we have to do with a first approximation only. The curves for all these diverse bodies excepting CS_2, while not identical, would form

TABLE XXII.

Ratios of Saturation-Pressures at constant fractions of the critical temperature to the Saturation-Pressures of Ethyl Oxide at the same fractions of its critical temperature.

	T_c.	Fractions of T_c.						
		·6.	·65.	·7.	·75.	·8.	·9.	1·0.
Acetone	506	·87	·90	·92	·94	·95	...	1·4
Methyl oxide......	404	1·7	1·5	1·4	1·4			
SO_2..................	428	1·7	1·7	1·7	1·7	1·7	...	2·2
NH_3	404	2·6	2·6	2·5	2·5	2·5	2·4	3·2
H_2S	373	...	3·6	3·2	2·8	2·6	...	2·6
CO_2	305	...	...	...	...	1·8	1·7	2·1
N_2O	308	...	...	...	...	2·4	1·8	2·1
CS_2..................	548	3·1	2·7	2·3	2·1	...	...	2·1
CCl_4	556	1·3	1·3	1·3	1·2	...	...	1·6
$CHCl_3$	537	1·6	1·5	1·5	1·4	1·3	...	1·6
CH_3Cl	404	1·7	1·5	1·4	1·4	...	...	2·1
C_2H_5Cl	456	1·4	1·4	1·3	1·2	1·2	...	1·5
C_2H_5Br	509	1·9	1·5	1·6	1·5	1·4		
Benzene, C_6H_6 ...	560	1·3	1·3	1·2	1·2	1·1	...	1·4

a compact bundle about a mean curve from which each body would have its own characteristic departure; and this is just what our study of B and β in Table XXI. should lead us to expect.

7. *Five Methods of finding the Virial Constant.*—The first method is that which we have already exhausted, namely, by means of extended enough observations of the compression and expansion of bodies in the gaseous state.

Second method: to obtain the virial constant l from one measure of the compressibility and of the expansibility at the same temperature of the body as a liquid.

Writing our infracritical equation thus,

$$p = \frac{R'T}{v}\left(1 + \frac{\sqrt{T}}{B} \cdot \frac{k'-v}{v-\beta}\right) - \frac{l}{2v^2},$$

$$\frac{\partial p}{\partial T} = \frac{R'}{v} + \frac{3}{2}\frac{R'\sqrt{T}}{B} \cdot \frac{k'-v}{v-\beta} = \frac{3}{2} \cdot \frac{l}{2v^2T} - \frac{R'}{2v} + \frac{3}{2}\frac{p}{T}.$$

But at ordinary low pressures the term p/T is a negligible part of this expression, and we can write

$$\frac{\partial p}{\partial \mathrm{T}} = \frac{3}{4} \cdot \frac{l}{v^2\mathrm{T}} - \frac{\mathrm{R}'}{2v}.$$

Now

$$\frac{\partial p}{\partial \mathrm{T}} = -\frac{\partial v}{\partial p} \Big/ \frac{\partial v}{\partial p} = -\frac{v_0}{v} \cdot \frac{\dfrac{1}{v_0}\dfrac{\partial v}{\partial \mathrm{T}}}{\dfrac{1}{v}\dfrac{\partial v}{\partial p}} = \frac{v_0}{v} \cdot \frac{a}{\mu},$$

where a and μ are the coefficients of expansion and the compressibility at T as usually defined.

$$\therefore \quad \frac{3}{4}\frac{l}{v^2\mathrm{T}} - \frac{\mathrm{R}'}{2v} = \frac{v_0}{v}\frac{a}{\mu};$$

$$\therefore \quad = \frac{4}{3}\left(v_0\frac{a}{\mu} + \tfrac{1}{2}\mathrm{R}'\right)v\mathrm{T} = \frac{4}{3}\left(v_0\frac{a}{\mu} + \tfrac{25}{26}\mathrm{R}\right)v\mathrm{T}.$$

In addition to giving us a value of l, this last equation gives a test as to whether the equation applies to a body or not, as the expression on the right-hand side is to be constant at all temperatures if μ is measured at low pressures. But on account of the experimental difficulties hitherto met with in the measurement of μ, the equation gives no very delicate test, although it might with the improvements in accuracy made within the last few years. In addition to ethyl oxide, the two substances for which we have measurements of both a and μ over the widest range of temperature are ethyl chloride, studied as to expansion by Drion and as to compression by Amagat, who has corrected Drion's coefficients of expansion for change of pressure, and pentane, studied by Amagat and Grimaldi. The following are the values of l calculated from the data at different temperatures for these two substances, with the megadyne taken as unit of force.

Temperature C. ...	0°.	11°.	13°.	20°.	40°.	60°.	80°.	100°.
C_5H_{12} (Grimaldi)...	8530	...	...	7460	7230	8000	8300	9250
C_5H_{12} (Amagat) ...	...	...	9200	...	...	...	...	9250
C_2H_5Cl	...	5820	...	...	...	5450	...	5270

This comparison has been made to show how, from measurements of liquid compressibility at present available, we can get only a fair idea of the value of l, but not an accurate measurement of it.

Amagat has determined the compressibility of several other liquids at different temperatures (*Ann. de Chim. et de Physique*, sér. 5, t. xi.) ; so have Pagliani and Palazzo (Wied. *Beibl.* ix.) and de Heen (Wied. *Beibl.* ix.), but their discussion would bring out nothing more than the above comparison has. De Heen's results would appear to make l diminish with rising temperature in every case ; but he measured his compressibilities in comparison with that of water at the same temperature, and to calculate their values used Pagliani and Vincentini's values for water (Wied. *Beibl.* viii.), which make the compressibility of water much more variable with temperature than Grassi's. If we used Grassi's values in de Heen's experiments, l would remain nearly constant. We will accordingly use the compressibility-method of calculating l subsequently, only to illustrate the general agreement of values derived from purely mechanical experiments with those found by the more accurate methods to which we now proceed. The values found by this second method are tabulated later on in Table XXIV.

Third method of finding the virial constant l: from the latent heat. If we differentiate with respect to T our equation for saturation-pressures (see Section 6),

$$\mathrm{P}(v_3-v_1)=\int_{v_1}^{v_3} p\,dv,$$

we get

$$\frac{d\mathrm{P}}{d\mathrm{T}}(v_3-v_1)+\mathrm{P}\left(\frac{dv_3}{d\mathrm{T}}-\frac{dv_1}{d\mathrm{T}}\right)=\int_{v_1}^{v_3}\frac{\partial p}{\partial \mathrm{T}}\,dv+\mathrm{P}\left(\frac{dv_3}{d\mathrm{T}}-\frac{dv_1}{d\mathrm{T}}\right);$$

$$\therefore\quad \frac{d\mathrm{P}}{d\mathrm{T}}(v_3-v_1)=\int_{v_1}^{v_3}\frac{\partial p}{\partial \mathrm{T}}\,dv.$$

Now from thermodynamics we have the relation

$$\mathrm{J}\lambda=(v_3-v_1)\mathrm{T}d\mathrm{P}/d\mathrm{T},$$

where λ is the latent heat,

$$\therefore\ \mathrm{J}\lambda=\int_{v_1}^{v_3}\mathrm{T}\frac{\partial p}{\partial \mathrm{T}}\,dv,$$

which of course could have been written down immediately, for if we write it in the form

$$\mathrm{J}\lambda=\int_{v_1}^{v_3}\left(\mathrm{T}\frac{\partial p}{\partial \mathrm{T}}-p\right)dv+\mathrm{P}v_3-\mathrm{P}v_1,$$

we remind ourselves that the latent heat of evaporation of a liquid is the heat supplied to neutralize a Thomson and Joule cooling effect.

We must evaluate the integral in three stages,

$$\int_{v_1}^{v_3}\mathrm{T}\frac{\partial p}{\partial \mathrm{T}}dv=\int_{k'}^{v_3}\mathrm{T}\frac{\partial p}{\partial \mathrm{T}}dv+\int_{k}^{k'}\mathrm{T}\frac{\partial p}{\partial \mathrm{T}}dv+\int_{v_1}^{k}\mathrm{T}\frac{\partial p}{\partial \mathrm{T}}dv,$$

using in each integral the equation that holds between its limits.

In the first,

$$pv=\mathrm{RT}\left(1+\frac{2k}{v+k}\right)-\frac{l}{v+k},$$

$$\frac{\partial p}{\partial \mathrm{T}}=\frac{\mathrm{R}}{v}\left(1+\frac{2k}{v+k}\right),$$

$$\mathrm{T}\frac{\partial p}{\partial \mathrm{T}}=p+\frac{l}{v+k}.$$

In the second,

$$pv=\mathrm{R'T}\left(1+\frac{\sqrt{\mathrm{T}}}{\mathrm{B}}.\frac{k'-v}{v-\beta}\right)-\frac{}{v+k},$$

$$\mathrm{T}\frac{\partial p}{\partial \mathrm{T}}=\frac{3}{2}\left(p+\frac{l}{v(v+k)}\right)-\frac{\mathrm{R'T}}{2v}.$$

In the third,

$$pv=\mathrm{R'T}\left(1+\frac{\sqrt{\mathrm{T}}}{\mathrm{B}}.\frac{k'-v}{v-\beta}\right)-\frac{l}{2v^2},$$

$$\mathrm{T}\frac{\partial p}{\partial \mathrm{T}}=\frac{3}{2}\left(p+\frac{l}{2v^2}\right)-\frac{\mathrm{R'T}}{2v}.$$

Hence

$$\mathrm{J}\lambda=\int_{k'}^{v_3}\left(p+\frac{l}{v+k}\right)dv+\int_{k}^{k'}\left\{\frac{3}{2}\left(p+\frac{l}{v+k}\right)+\frac{\mathrm{R'T}}{2v}\right\}dv$$

$$+\int_{v_1}^{k}\left\{\frac{3}{2}\left(p+\frac{l}{2v^2}\right)-\frac{\mathrm{R}'}{2v}\right\}dv$$

$$=\int_{v_1}^{v_3} pdv + \frac{1}{2}\int_{v_1}^{k'} pdv + \int_{k}^{v_3}\frac{ldv}{v(v+k)} + \frac{1}{2}\int_{k}^{k'}\frac{ldv}{v(v+k)}$$

$$+\frac{3}{4}\int_{v_1}^{k}\frac{ldv}{v^2} - \frac{1}{2}\int_{v_1}^{k'}\frac{\mathrm{R'T}dv}{v}$$

$$=\mathrm{P}(v_3-v_1) + \frac{1}{2}\int_{v_1}^{k'} pdv + \frac{l}{k}\log\frac{2v_3}{v_3+k} + \frac{l}{2k}\log\frac{2k'}{k'+k}$$

$$+\frac{3l}{4}\left(\frac{1}{v_1}-\frac{1}{k}\right) - \frac{\mathrm{R'T}}{2}\log\frac{k'}{v_1}.$$

We must now evaluate the only integral that has been left unevaluated,

$$\int_{v_1}^{k'} pdv = \int_{v_1}^{k'}\frac{\mathrm{R'T}}{v}\left(1+\frac{\sqrt{\mathrm{T}}}{\mathrm{B}}\cdot\frac{k'-v}{v-\beta}\right)dv - \int_{k}^{k'}\frac{ldv}{v(v+k)} - \int_{v_1}^{k}\frac{ldv}{2v^2}$$

$$=\mathrm{R'T}\log\frac{k'}{v_1} + \frac{\mathrm{R'T}\sqrt{\mathrm{T}}}{\mathrm{B}}\left(\frac{k'-\beta}{\beta}\log\frac{k'-\beta}{v_1-\beta} - \frac{k'}{\beta}\log\frac{k'}{v_1}\right)$$

$$-\frac{l}{k}\log\frac{2k'}{k'+k} - \frac{l}{2}\left(\frac{1}{v_1}-\frac{1}{k}\right).$$

These last expressions are the only ones that introduce B and β into the value of Jλ, and it is desirable to remove these two constants. At low pressures we can neglect the *pv* term in the infracritical equation, and we get

$$\frac{\mathrm{R'T}\sqrt{\mathrm{T}}}{\mathrm{B}} = \left(\frac{l}{2v_1} - \mathrm{R'T}\right)\frac{v_1-\beta}{k'-v_1};$$

removing B by means of this we get

$$\mathrm{J}\lambda = \mathrm{P}(v_3-v_1) + \frac{l}{k}\log\frac{2v_3}{v_3+k} + \frac{l}{2}\left(\frac{1}{v_1}-\frac{1}{k}\right)$$

$$+\frac{1}{2}\left(\frac{l}{2v_1} - \mathrm{R'T}\right)\frac{v_1-\beta}{k'-v_1}\left(\frac{k'-\beta}{\beta}\log\frac{k'-\beta}{v_1-\beta} - \frac{k'}{\beta}\log\frac{k'}{v_1}\right).$$

We can now remove β and greatly simplify this equation, if we apply it to latent heats near the ordinary boiling-point T_b. The last term taken in its entirety has a small numerical value compared to the rest, so that in it we can make approximations without any sacrifice of accuracy worth considering: we have seen that β is approximately proportional to k (see Table XXI.) and v_1, the volume of the liquid at the boiling-point, may be assumed to be approximately proportional to k' the volume at the critical temperature, and $k'=7k/6$: hence the coefficient of $(l/2v_1-\mathrm{R}'\mathrm{T})$ in the last term is approximately the same for all bodies and we can evaluate it for ethyl oxide ; call it C. Again, in $\mathrm{P}(v_3-v_1)$ neglect v_1 and assume the gaseous law $\mathrm{P}v_3=\mathrm{RT}_b$. And further, k is small compared to v_3, so that $2v_3/(v_3+k)$ is nearly 2, and its value for ethyl oxide can be applied to all bodies. $\mathrm{R}'=25\mathrm{R}/13$. Hence multiplying by M the molecular weight we can write

$$\frac{\mathrm{M}l}{k}\left\{\frac{1}{2}\left(\frac{k}{v_1}-1\right)+\log\frac{2v_3}{v_3+k}+\frac{\mathrm{C}}{2}\frac{k}{v_1}\right\}=\mathrm{JM}\lambda+\mathrm{MR}\left(\frac{25}{13}\mathrm{C}-1\right)\mathrm{T}_b.$$

MR is the same for all bodies, and T_b is the absolute boiling-point. This equation still involves k as well as l ; when k is not known we must eliminate it by means of our previous assumption, namely, that k is proportional to v_1, which we know to be approximately true ; in so far as it is inexact it will introduce inexactness into our calculation of l. Accordingly in symbols $k/v_1=r$, where r is the same for all bodies, and can be found for ethyl oxide. Making the numerical reductions we get

$$\mathrm{M}l/v_1=66{\cdot}5\,\mathrm{M}\lambda-101\mathrm{T}_b$$

as the equation which gives l in terms of the megadyne as unit of force, when λ is the latent heat of a gramme in calories and v_1 its volume in cubic centimetres at the absolute boiling-point T_b. This equation will be abundantly verified afterwards in Table XXIV.; but meanwhile, if to test it we apply it to calculate the latent heat of ethyl oxide, we find $\lambda=83{\cdot}4$, whereas several experimenters have agreed in an estimate of about 90 ; but, on the other hand, Ramsay and Young (Phil. Trans. 1887) have made a special study of the terms in the thermodynamic relation $\mathrm{J}\lambda=(v_3-v_1)\mathrm{T}d\mathrm{P}/d\mathrm{T}$, and have so calculated values of λ almost up to the critical temperatures, their value at the boiling-point is 84·4, and there is the same amount of discrepancy between their values at higher temperatures and Regnault's experimental determinations. Yet Perot, who has made an elaborate study (*Ann. de Ch. et de Ph.* sér. 6, t. xiii.) both of λ experimentally and of the quantities involved in its calculation by the thermodynamic relation, has found the most perfect harmony between the results of the two methods. Now at 30° C. Perot gives as the saturation-volume of the vapour 400·4, and the saturation-pressure ·635 metre, while Ramsay and Young's values are 374 and ·648 metre; but if ethyl oxide were a perfect gas, under Perot's pressure of ·635 metre it would have a volume of 400·8, almost identical with his value: yet we cannot imagine that ethyl oxide under this pressure and at this temperature is so nearly a perfect gas as this would imply, unless there is some remarkable discontinuity in its behaviour at high volumes. Accordingly, in spite of the thoroughness of the researches of both Perot and Ramsay and Young, we are on the horns of a triple dilemma, from which only some experimental repetition can deliver us, and demonstrate where the cause of these discrepancies lies. Wüllner and Grotrian (Wied. *Ann.* xi.) have put on record the results of experiments which indicate the cause; they find the pressure of condensation measurably different from the ordinarily measured saturation-pressure,—a fact explaining the difficulty of measuring v_3 accurately, and showing also that the values of dP/dT are not so reliable as usually supposed.

Our last equation is verified by, and shows us the cause of, an interesting relation that has been independently discovered and expressed in different forms, between the molecular latent heat and the boiling-point, by Pictet (*Ann. de Ch. et de Ph.* sér. 5, t. ix. 1876), Trouton (Phil. Mag. xviii. 1884), and Ramsay and Young (Phil. Mag. xx. 1885), namely, that the molecular latent heats of fluids are nearly proportional to their absolute boiling-points. Now we have seen that $T_c = 120l/409Rk$ (Section 4), and I have noticed that a large number of substances have their ordinary absolute boiling-points nearly equal to $2T_c/3$, and k is nearly proportional to v_1, say is equal to $2{\cdot}8\,v_1$, as it is for ethyl oxide. Hence we have

$$\frac{3}{2}T = \frac{120\,l}{409\,R\,2{\cdot}8\,v_1};$$

$$\therefore\ Ml = 14{\cdot}3\,M\,R\,v_1 T_b = 1190\,v_1 T_b,$$

when the megadyne is the unit of force; hence from our equation for Ml in terms of λ we have

$$1190T_b = 66{\cdot}5\,M\lambda - 101\,T_b\,;$$

$$\therefore\ 1291\,T_b = 66{\cdot}5\,M\lambda \quad \text{or} \quad M\lambda = 19{\cdot}4T_b,$$

or the molecular latent heat is proportional to the absolute boiling-point.

(It is to be noted that $Ml = 1190\,v_1T_b$ gives a rough means of obtaining l from the boiling-points and the volumes at the boiling-points of liquids, which might be convenient when better data are wanting.)

Robert Schiff (*Ann. der Chem.* ccxxxiv.) has made the most accurate determinations to test this relation between molecular latent heat and boiling-point. For 29 compounds of the form $C_nH_{2n}O_2$, from ethyl formiate up to isoamyl valerate, he finds $M\lambda = 20{\cdot}8T_b$ in the mean, the greatest departures being 20·4 for propyl isobutyrate, and 21·1 for propyl formiate; for 8 hydrocarbons of the benzene series he finds a mean coefficient 20, with 19·8 for cymene and 20·6 for benzene as the greatest departures. To these 37 examples we will add the following from Trouton's paper, doubling his numbers, as he used density instead of M.

TABLE XXIII.—Values of $M\lambda/T_b$.

C_2H_5Cl.	$CHCl_3$.	CCl_4.	$AsCl_3$.	$SnCl_4$.	SO_2.	CS_2.
21	22	21	21	20	23	21

$(C_2H_5)_2O$.	$(C_5H_{11})_2O$.	$(CH_3)_2CO$.	$C_{10}H_{16}$.	$(C_2H_5)_2C_2O_4$.
22	24	23	22	23

The mean value of the coefficient is higher than that deduced theoretically above (19·4), because in round fraction we wrote $T_b = 2T_c/3$, but the general truth of the relation is well enough brought out.

Fourth method of finding the virial constant l: from the critical temperature and pressure. Now we have (Section 4),

$$T_c = 120l/409Rk, \quad p_c = 36l/409k^2,$$

$$\therefore\ T_c/p_c = 10k/3R \quad \text{and} \quad l = 409R^2T_c^2/400p_c,$$

this is for compounds; for elements $l = 27R^2T_c^2/64p_c$. Where both the critical pressure and temperature are known, this gives l theoretically with accuracy, but practically the difficulties in measuring the critical pressure introduce inaccuracy. In the relation $T_c/p_c = 10k/3R$, as R varies inversely as the molecular weight, we see that the molecular domains (Molecular volumes) of bodies at the critical temperature are proportional to the quotient of critical temperature by critical pressure, a relation which Dewar has proved experimentally (Phil. Mag. xviii. 1884) for 21 volatile bodies, for which he has determined and collected the data. These with other data since published enable us to determine values of l for certain bodies for which the other methods are not available.

As there are many more critical temperatures determined up to the present than critical pressures, and as we have seen that an error in the critical temperature is of less relative importance than an error in the critical pressure, we can make ourselves independent of critical pressures with advantage, by employing the approximation that has already been useful to us, that k is proportional to the volume of the liquid at the boiling-point or $k = 2{\cdot}83v_1$. Then

$$Ml = 409MRT_ck/120 = 800T_cv_1$$

approximately, with the megadyne as unit of force. This is a more accurate form of the relation $Ml = 1190T_bv_1$ given above, in which we assumed the approximation $T_b = 2T_c/3$.

8. *Fifth or Capillary Method of finding the Internal Virial Constant*, with digressions on the Brownian movement in liquids and on molecular distances.—So far we have been proceeding on a purely inductive path, with two deductive guides in Clausius's equation of the virial and in the law of the inverse fourth power, which requires that the internal virial should vary inversely as the volume. But now, in passing on to our fifth and most useful method of finding l

from surface-tension, we must employ a deductive relation between l and surface-tension, furnished by the law of the inverse fourth power. In a previous paper (Phil. Mag. July 1887) it was shown that if the law of force between two molecules of mass m, at distance r apart is $3\text{A}m^2/r^4$, then the internal virial for the molecules in unit mass is $3\pi\text{A}\rho \log \text{L}/a$, ρ being density, and L a finite length of the order of magnitude of the linear dimensions of the vessels used in physical measurements, a being the mean distance apart of the molecules. The ratio L/a remains the same for a given mass whatever volume it occupies, but I also assumed that L/a is so large a number that $\log \text{L}/a$ would hardly be affected by such large variations as might occur in the value of L when the behaviour of a kilogramme of a substance was compared with that of a milligramme. To remove the haziness of this assumption, I will now make a more accurate evaluation of the internal virial.

By definition it is $\frac{1}{2}\cdot\frac{1}{2}\cdot\Sigma\Sigma 3\text{A}m^2/r^3$, and we will evaluate it for a spherical mass. To cast it into the form of an integral, take any molecule m amongst the number n in a spherical vessel of radius R; gather it to its centre as a true particle and spread the remaining $n-1$ in a uniform continuous mass separated from m by a small spherical vacuum of radius a, so chosen that the virial of m and the continuous mass is the same as that of m and the $n-1$ molecules.

Suppose m at the point O, and the centre of the vessel at C, and let $\text{OC}=c$. Take OC as axis of x and any two rectangular axes through O as axes of y and z. Let polar coordinates $r\,\theta\,\phi$ be related to these in the usual manner. Then the equation to the surface of the sphere is

$$(x-c)^2+y^2+z^2=\text{R}^2 \text{ or } r^2-2cr\sin\theta\cos\phi+c^2-\text{R}^2=0.$$

Let r_1 and r_2 be the two roots of this equation in r so that $r_1r_2=\text{R}^2-c^2$. Then m^2/r^3 can be replaced by

$$m\rho r^2\sin\theta\, d\theta\, d\phi\, dr/r^3,$$

and

$$\Sigma m^2/r^3 \text{ by } \iiint m\rho\sin\theta\, d\theta\, d\phi\, dr/r.$$

If we integrate with respect to r on one side of the plane yz from a to r_1 and on the other from a to r_2 and add the two results, then we have to take θ and ϕ each between 0 and π, thus:

$$\Sigma m^2/r^3 = \int_0^\pi \int_0^\pi m\rho \sin\theta d\theta\, d\phi \left[\int_a^{r_1} dr/r + \int_a^{r_2} dr/r\right].$$

The two integrals in brackets give

$$\log r_1 r_2/a^2 = \log (\mathrm{R}^2 - c^2)/a^2.$$

Hence

$$\Sigma m^2/r^3 = 2\pi m\rho \log (\mathrm{R}^2 + c^2)/a^2.$$

To perform the second summation we can first add the values of the last expression for all the molecules at distance c from the centre of the vessel, and write the result in the form

$$2\pi\rho 4\pi\rho c^2 dc \log (\mathrm{R}^2 - c^2)/a^2,$$

and we then have

$$\tfrac{1}{2} \cdot \tfrac{1}{2}\Sigma\Sigma 3\mathrm{A}m^2/r^3 = 6\mathrm{A}\pi^2\rho^2 \int_0^{\mathrm{R}-a} c^2 dc \log (\mathrm{R}^2 - c^2)/a^2.$$

Evaluating the integral, this becomes

$$6\mathrm{A}\pi^2\rho^2 \left\{ \frac{1}{3}(\mathrm{R}-a)^3 \log\frac{2\mathrm{R}-a}{a} - \frac{2}{9}(\mathrm{R}-a)^3 - \frac{2}{3}\mathrm{R}^2(\mathrm{R}-a) + \frac{\mathrm{R}^3}{3}\log\frac{2\mathrm{R}+a}{a} \right\},$$

in which, neglecting unity in comparison with the large number R/a, we get

$$\mathrm{A}\pi^2\rho^2\mathrm{R}^3\{4 \log 2\mathrm{R}/a - 16/3\}.$$

But if W is the total mass in the vessel, then

$$\mathrm{W} = 4\pi r^3\rho/3,$$

and we get

$$\mathrm{W}a\pi\rho\{3 \log 2\mathrm{R}/a - 4\}.$$

When $\mathrm{W}=1$ the first term of this becomes identical with the value of the internal virial previously given, with 2R written instead of L. Replacing R by its value in terms of W and ρ, we get for the internal virial of mass W,

$$\mathrm{WA}\pi\rho(\log 6\mathrm{W}/\pi\rho a^3 - 4).$$

As ρa^3 is constant, we see that for a given mass the internal virial for molecular force varying inversely as the fourth power is rigorously proportional to the density, but it is not purely proportional to the mass. Although the number $6\mathrm{W}/\pi\rho a^3$ is a large one, and has a logarithm varying slowly with W, yet large enough variations in W can affect it appreciably, as we see if provisionally we accept Sir W. Thomson's estimate of 2×10^{-9} centim. as the lowest possible value for a. Suppose $\pi\rho = 3$, then $6W/\pi\rho a^3$ is $10^{27}W/4$, and if W is 4000, 4, or ·004 grm., then the values of log $6W/\pi\rho a^3$ are as 30, 27, and 24, and we have a larger mass variation of the internal virial than is likely to have escaped detection in its effects, such as a difference in the density, expansion, compressibility, latent heat, and saturation-pressures of a liquid as measured on a milligramme, from their values as measured on a kilogramme. The raising of this difficulty suggests to us in passing that there exists a department of microphysics in which little has as yet been done by the experimenter, and that great interest would attach to a research determining when a mass variation of the properties usually spoken of as physical constants actually sets in.

But meanwhile we must scrutinize more closely the meaning of our last result. According to the views of Laplace (and of the early elasticians), if a plane be drawn dividing a mass of solid or liquid into two parts, then, in consequence of molecular force, the one part exercises a resultant attraction on the other, and this has to be statically equilibrated by a pressure (called the molecular or internal pressure) acting across the plane, a conception which is necessary in any purely statical theory of elasticity. Adopting for the moment this mode of viewing things, we see that our result amounts to this, that the internal pressure is measurably greater at the centre of a kilogramme than of a milligramme.

But if we try to carry out the kinetic theory in its integrity, we must reject the idea of a statical pressure, and replace it by its kinetic equivalent of a to and fro transfer of momentum ; this may take place as a quite indiscriminate traffic of individual molecules across the plane, or as such a traffic modified by the existence of streams of molecules in opposite directions. If streams, or motions of molecules in swarms, actually exist in fluids, then our interpretation of the equation of the virial would have to take account of their existence. The kinetic energy of the motion of a swarm as a whole would not count as heat but as mechanical energy, and for the amount of it we should have approximately the kinetic energy of the swarm motion equal to the virial of the forces between the swarms.

Therefore we require to divide the energy into two parts, that of molecular motion inside the swarm constituting heat and that of the swarm ; in the same way the internal virial is divided into two parts, one within each swarm, the other between the swarms. But the swarms could on the average be regarded as equivalent to spheres of radius L, where L must be supposed nearly independent of mass and liable to the same variations with temperature and pressure as the linear dimensions of any quantity of the liquid, so that L is proportional to a, and our expression (theoretical) $\mathrm{WA}\pi\rho(3\log 2\mathrm{L}/a-4)$ for the internal virial becomes purely proportional to the mass and purely proportional to the density, as $3\mathrm{W}l/4v$ our experimental internal virial for a mass W of a compound liquid is.

This hypothesis would affect somewhat the rigorousness of certain thermodynamical relations as usually interpreted, such as $\mathrm{J}\lambda=(v_3-v_1)\mathrm{T}dp/d\mathrm{T}$, since it provides a supply of internal mechanical energy not taken account of ; but if this supply is only slightly variable with pressure and temperature it would make little difference in most parts of thermodynamics.

With the addition of this hypothesis of molecular swarms, which will be used only in calculating molecular distances, and will not affect at all the rest of our work, the law of the inverse fourth power is brought into strict harmony with the behaviour of compound liquids and of elements both as liquids and gases. We must therefore inquire what experimental evidence there is for the existence in liquids of a motion of swarms of molecules, possessing the remarkable property of not being degraded to heat as ordinary visible motions are. In the motion long familiar to microscopists as the Brownian movement we have such evidence. Gouy has recently (*Compt. Rend.* cix. p. 103) recalled the attention of physicists to this remarkable ceaseless motion of granules in liquids. He states that it occurs with all sorts of granules, and with an intensity less as the liquid is more viscous and as the granules are larger. It occurs when every precaution is taken to ensure constant temperature, and to ensure the absence of all external causes of motion. Granules of the same size but as different in character as solid granules, liquid globules, and gaseous bubbles, show but little difference in their motions—a fact which proves that the cause is to be looked for not in the granules themselves but in the liquid, the granules being merely an index of motions existing in the liquid. The most pronounced character of the motion is its rapid increase with diminishing size of granules, so that all that is seen under the microscope is the limit of movements of unknown magnitude. Gouy considers the Brownian movements to be a remote result of the motion of the molecules themselves, but according to what we know of molecular dimensions I fancy that the Brownian movement must be considered rather as a sign of the motion of swarms of molecules. If swarms of molecules are weaving in and out amongst one another, so that the average transfer of momentum at a point is the same in all directions, then the vibratory agitation of granules amongst the swarms is just what we should expect. The striking fact about the Brownian movement is that it is ceaseless; it is never degraded into heat. This alone forces us to conceive a form of motion existing ~~in~~ liquids on a larger scale than molecular motion but possessing its character of permanence; in other words, the motion of swarms of molecules.

The existence of swarms would not affect our views of the rise of liquids in capillary tubes as a purely statical question; so that, for the connexion between molecular force and surface-tension, we can use the calculation given in another paper (Phil. Mag. April 1889) (rather badly affected with misprints), where I have shown that the surface-tension of liquids that wet glass, measured in tubes so narrow that the meniscus-surface is a hemisphere, is given by the equation

$$\alpha=\pi\bar{\rho}^2\mathrm{A}e/(2+\sqrt{2})\,;$$

where ρ is the average density of the capillary surface-film (to be written also $1/\bar{v}$), and e is the distance which we must suppose to be left between a continuous meniscus and the base of a continuous column raised by its attraction, if the action between the continuous distributions is to be the same as in the natural case of discontinuous molecular constitution of meniscus and column. The distance e is not identical with the length a which occurs in our theoretical value of the internal virial of unit mass,

$$\mathrm{A}\pi\rho(3\log 2\mathrm{L}/a-4)=\frac{3}{2}\,\frac{l}{2v},$$

but it is closely proportional to it. If we can find the relation between e and a, then from capillary determinations we can obtain relative values of the virial constant l which, as we have already found some absolute values of l, can be converted to absolute values; at the same time, too, we shall be able to find a value of a the mean distance apart of the molecules.

To find the relations between e and a we can proceed thus. If we have a single infinite straight row of molecules at a distance a apart, the force exerted by one half of it on the other is

$$\Sigma_{n=0}^{n=\infty}\Sigma_{p=1}^{p=\infty}\frac{3\mathrm{A}m^2}{(p+n)^4a^4},$$

which can easily be evaluated as approximately $3\cdot 6\,\mathrm{A}m^2/a^4$. Two infinite continuous lines in the same line, of density m/a with distance e between their contiguous ends, would exert a force

$$\frac{3\mathrm{A}m^2}{a^2}\int_e^\infty\int_0^\infty\frac{dx\,dy}{(x+y)^4}$$

on one another; this is equal to $\mathrm{A}m^2/2a^2e^2$. If, then, the continuous distribution is to be equal to the molecular, we have

$$e^2=a^2/7\cdot 2,\quad e=a/2\cdot 7.$$

Again, if along two infinite axes one at right angles to the other and terminating in an origin O at its middle point molecules are arranged along each at distance a apart starting from O, then the force exerted by the unlimited row on the other is

$$2\Sigma_{p=1}^{p=\infty}\Sigma_{n=1}^{n=\infty}\frac{3\mathrm{A}m^2n}{(p^2+n^2)^{\frac{5}{2}}a^4}+\Sigma_{p=1}^{p=\infty}\frac{3\mathrm{A}m^2}{p^4a^4},$$

which can be evaluated at about $5\mathrm{A}m^2/a^4$.

Replace the rows by two continuous line-distributions of density m/a, the one terminating at a distance e from O: it is required to find e so that the force may be the same as this. The force is

$$\frac{6\mathrm{A}m^2}{a^2}\int_e^\infty\int_0^\infty\frac{y\,dy\,dx}{(x^2+y^2)^{\frac{5}{2}}}=\mathrm{A}m^2/a^2e^2.$$

Hence in this case

$$e^2=a^2/5, \quad e=a/2{\cdot}2.$$

From these two simple cases we get an idea of the relation between e and a. The case of a meniscus attracting the column which it raises in a capillary tube is more analogous to the second than to the first, and it is easy to see that in the case of the meniscus we can say that e is not less than $a/2{\cdot}2$. It will suffice to write $e=a/2{\cdot}2$.

Now according to the definition of a in our theoretical expression for the internal virial, it is the radius of the spherical vacuum artificially used to represent the domain of a molecule; but as it occurs in the expression $\log 2\mathrm{L}/a$, where $2\mathrm{L}/a$ is a very large number and the value of L is indefinite, we see that there is no inaccuracy in making it identical with a the mean distance apart of the molecules. However, for the sake of formal completeness, we can easily find the relation between the two quantities which we have denoted by the one symbol a. Let us now denote by x the mean distance apart of the molecules, that is the edge of the cube in a cubical distribution of the molecules; then, from the definition of a, x and a are connected by the relation

$$\Sigma \mathrm{A}m^2/r^3=\int_a^{\mathrm{R}} 4\pi \mathrm{A}m^2 dr/x^3 r,$$

the summation being extended to all the molecules in a sphere of radius R. By actual summation up to $\mathrm{R}=5x$ we find approximately $a={\cdot}9x$.

With our previous estimate of e as $a/2{\cdot}2$, which we must now write $x/2{\cdot}2$ on account of our change of symbol for mean distance apart, we have the two equations,

$$l=\mathrm{A}\pi(4\log 2\mathrm{L}/{\cdot}9x-16/3),$$

$$\alpha=\pi\bar{\rho}^2\mathrm{A}x/2{\cdot}2(2+\sqrt{2}).$$

We can replace $\bar{\rho}$ by ρ, the difference between them being necessarily very slight. Then for ethyl oxide we have the following data: α at the boiling-point according to Schiff is 1·57 grammes weight per metre, or $1{\cdot}57/10^6$ kilog. per cm.;

l is 7500 kilog. cm.^{4}, and $v_1 = 1{\cdot}44$ cm.^{3} Eliminating A from the two equations, we have a relation between x and L, namely

$$x = 7{\cdot}5 v_1^2 \frac{\alpha}{l} (9{\cdot}2 \log_{10} 2L/{\cdot}9x - 16/3).$$

L being hypothetical is not known to us, but we can give it a series of possible values, and calculate by trial from the last equation the corresponding series of values for x, with the following results:—

L.	x.
$1/10^5$ cm.	$4{\cdot}6/10^7$ cm.
$1/10^4$ „	7·7 „ „
$1/10^3$ „	11·0 „ „

As it is subsequently to be shown that in liquid ethyl oxide and in all regular compound liquids the molecules are paired, and that each pair acts on the others as if it were a single molecule, we may estimate it as likely that the mean distance apart of the pairs in ethyl oxide is between 1 and 10 micromillimetres ($1/10^6$ mm.). This result, though 100 times as large as Sir William Thomson's limits for the distance apart of molecules in liquids, namely 2×10^{-9} cm. and 7×10^{-9} cm., is yet in better agreement with the estimate of molecular distances arrived at by Rücker (Journ. Chem. Soc. 1888) as the most probable result obtainable from the most important attempts yet made to measure the range of molecular forces. The most suggestive of these is Reinold and Rücker's discovery, that the equilibrium of a soap-solution film becomes unstable when its thickness is reduced to between 96 and 45 micromillimetres, but again becomes stable when the thickness is still further reduced to 12 micromillimetres. At the latter thickness the film shows black in reflected light. If the intermolecular distances are nearly the same in soap-solutions as in liquid ethyl oxide, then the black film must be regarded as consisting of a single layer of molecules or groups of molecules (in the case of water the molecules will subsequently be shown to go in double pairs). This is an intelligible result, and gives the simplest explanation of Reinold and Rücker's beautiful discovery of a stable thickness supervening on the unstable, for we recognize a single layer of molecules as a stable configuration. Of course it is to be understood that what we mean by the thickness of a single layer of molecules is the one *n*th part of the thickness of *n* layers ; and if the black film is really only a single layer, it is in this sense that Reinold and Rücker's estimate of 12 micromms. is to be taken, for they did not measure an actual distance from the front to the back of a black film, but only estimated from accurate and accordant measurements, made in entirely different manners, that the number of layers in the black film is to the number in a thickness of 1 centim. as 12 micromms. is to 1 centim.

If the black film consists really of only a single layer of molecules, it is surely a hopeful sign for molecular physics that measurements should have been possible on it, though only visible through its invisibility.

If, encouraged by this experimental support, we say that in round numbers the mean distance between the pairs of molecules in liquid ethyl oxide is 10 micromms., then one gramme contains $2v_1 \times 10^{18}$ molecules, or the mass of a single molecule is

$$1/2{\cdot}88 \times 10^{18} \text{ grm.} = 3{\cdot}5/10^{19},$$

and so the mass of an atom of H is

$$3{\cdot}5/74 \times 10^{19} = 5/10^{21} \text{ grm. nearly.}$$

It would lead us too far from our present purpose to discuss other estimates of molecular distance, especially as Reinold and Rücker's measurement of the black film is the most definite and striking yet made of these minute distances ; but the question of the range of molecular force is of special importance to us.

Quincke (Pogg. *Ann.* cxxxvii.) determined what thickness of silver it is necessary to deposit on glass so that the capillary effect on water may be the same as that of solid silver; that is, at what distance the difference between the molecular attractions of glass and silver for water becomes too small to be measured. He found the thickness to be about 50 micromms.

Now, according to the law of the inverse fourth power, the attraction of a cylinder of radius c, length h, and density ρ on a particle of mass m on the axis at a distance z from the nearest end is easily calculated as

$$2\mathrm{A}m\rho\pi\left(\frac{1}{z}-\frac{1}{z+h}-\frac{1}{\sqrt{c^2+z^2}}+\frac{1}{\sqrt{c^2+(z+h)^2}}\right).$$

If the cylinder consists of a length h_1 of silver with a length h_2 of glass, the silver being near the particle, then, the suffixes 1 and 2 applying to silver and glass, the attraction of the composite cylinder is

$$2\mathrm{A}_1 m\rho_1\frac{\pi}{z}-2m\pi(\mathrm{A}_1\rho_1-\mathrm{A}_2\rho_2)\left(\frac{1}{z+h_1}-\frac{1}{\sqrt{c^2+(z+h_1)^2}}\right)$$

$$-2\mathrm{A}_1 m\rho_1\frac{\pi}{\sqrt{c^2+z^2}}-2\mathrm{A}_2 m\rho_2\pi\left(\frac{1}{z+h_1+h_2}-\frac{1}{\sqrt{c^2+(z+h_1+h_2)^2}}\right).$$

Making the circumstances correspond to Quincke's experiment, we have z nearly equal to the mean molecular distance in water, about 10 micromms.; h_1 is small compared to h_2 and c, and, according to Quincke, is 50 micromms. when the composite cylinder exerts the same force on m as if it were all silver; accordingly the last expression reduces to the two terms

$$2\mathrm{A}_1 m\rho_1\pi/z-2m\pi(\mathrm{A}_1\rho_1-\mathrm{A}_2\rho_2)/(z+h_1),$$

which permit us to compare the molecular force range h_1 with the molecular distance z. That the second of these terms should become negligible when h_1 is 50 micromms. is a result quite in accordance with the value 10 micromms. for z.

Let us briefly compare the magnitudes of molecular and gravitational force. The most convenient plan will be to compare the two forces in the case of two single ethyl-oxide molecules at a distance of one centim. apart; that is, to calculate $\mathrm{A}m^2$ and $\mathrm{G}m^2$, where m is the actual mass of the molecule, and G the constant in the expression $\mathrm{G}m^2/r^2$ for gravitation.

In the expression

$$\alpha=\pi\bar{\rho}^2\mathrm{A}x/2\cdot 2(2+\sqrt{2}),$$

using the value 10 micromms. for x and the values previously given for the other quantities, we can find A, and then using the value $3\cdot5/10^{19}$ for the mass of a molecule of ethyl oxide we find $Am^2=9/10^{30}$ in terms of the dyne. To calculate G we have 981 as the acceleration of gravitation; the mass of the earth is 6×10^9 grm. and its radius is $6\cdot37\times10^7$cm., so that $Gm^2=2\cdot1/10^{29}$ in terms of the dyne. Hence at a distance of 1 centim. the gravitation of two ethyl-oxide molecules is about double their molecular attraction, or, allowing for uncertainties in our calculation, we may say that at about 1 centim. apart two molecules exert the same gravitational as molecular force on one another.

We now return to the main business of this section, which is the *Fifth Method* of finding the virial constant l. This consists in using the equation already used for calculating molecular distances in the form

$$l=7\cdot5v^2\alpha(4\log 2L/\cdot9x-16/3)\Big/x.$$

Now x, the molecular distance for different liquids, varies as $m^{\frac{1}{3}}v^{\frac{1}{3}}$, and the expression in brackets may be assumed to be the same for all bodies; hence $l=c\alpha v^{\frac{5}{3}}/m^{\frac{1}{3}}$, where c is a constant whose value can be obtained on substituting in the case of ethyl oxide the known values of l, α, v, and m, or, more safely, by taking a mean value from several substances.

But we must remember that we are using v the volume in the body of the liquid, instead of $\bar{v}$ that in the surface-film; a replacement which is not justified by experiment, seeing that for a given liquid $\alpha v^{\frac{5}{3}}$ measured at different temperatures is not constant, the reason being that $\bar{v}$ varies much more rapidly with temperature than v. But, in our ignorance of the relation between surface and body-density, all that we seek for from the above equation for l, is true values for l from measured values for α. Accordingly the question arises, Can we choose temperatures at which to measure α for different substances, so as to get true relative values of l irrespective of our ignorance of $\bar{v}$?

As we have seen (Section 6) that at equal fractions of their critical temperatures, and under equal fractions of their critical pressures, one liquid is approximately a model of another on a different scale, we conclude that if we use the value of the surface-tension measured at a constant fraction of the critical temperature, and under a constant fraction of the critical pressure, we ought to get correct relative values of l; as surface-tension is not appreciably affected by pressure, we can dispense with the condition as to pressure and use measurements of α made under a pressure of one atmosphere at a constant fraction of the critical temperature. I have chosen the fraction as two-thirds, because it gives a temperature near to the boiling-point of most liquids.

Schiff's abundant measurements (*Ann. der Chem.* ccxxiii., and, further, Wied. *Beibl.* ix.) include not only the height to which different liquids rise in a capillary tube at their boiling-points, but also its temperature-coefficient, which is such as to show that the height in every case vanishes near the critical point.

Let H be the height to which a liquid rises in a tube of radius 1 millim.; then if H really vanished at the critical temperature and varied linearly with temperature, we should at $2T_c/3$ have $H=T_c b/3$, where b is the temperature-coefficient. But to use this would be to depend too much on the accuracy of b.

If H_b is the value at T_b the boiling-point, then $T_c=T_b+H_b/b$, and

$$H=H_b+(T_b-2T_c/3)b=H_b/3+T_b b/3,$$

which depends partly on H_b, measured by Schiff, and partly on b. Now $\alpha=H\rho/2=H/2v$;

$$\therefore\ l \text{ or } c\alpha v^{\frac{4}{3}}/m^{\frac{1}{3}}=cHv^{\frac{1}{3}}/2m^{\frac{1}{3}},$$

$$l=c(H_b/3+T_b b/3)v^{\frac{1}{3}}/2m^{\frac{1}{3}}.$$

If H is measured according to the usual practice as the height in millimetres for a tube of radius 1 millim., that is, if α is measured in grammes weight per metre, then if l is desired in terms of the megadyne, gramme, and centimetre as units, $c/2=5930$, a mean value. Apart from all hypothesis about molecular force, our last relation between the virial constant and the constants of capillarity will be amply confirmed by the extensive comparisons soon to be presented in

Tables XXIV. and XXV. Meanwhile a few consequences of the relation may be glanced at.

9. *Establishment on Theoretical grounds of Eötvös's relation between surface-tension, volume, and temperature.*—According to the modified equation of the fourth method of finding l, $Ml=800T_c v_1$; and according to that of the fifth method, $l=c\alpha v^{\frac{5}{3}}/m^{\frac{1}{3}}$. The first of these equations would be more accurate if we replaced v_1 by v, which in the second means the volume at $2T_c/3$; so $Ml=800T_c v$, and m the actual mass of the molecule is proportional to M its molecular weight; so that from the second we have Ml proportional to $\alpha(Mv)^{\frac{2}{3}}v$, and hence $\alpha(Mv)^{\frac{2}{3}}$ measured at $2T_c/3$ is for all bodies proportional to T_c. Now in our notation the relation discovered by Eötvös (Wied. *Ann.* xxvii.) is

$$d\{\alpha(Mv)^{\frac{2}{3}}\}/dT=\cdot 227,$$

or

$$\alpha(Mv)^{\frac{2}{3}}=\cdot 227(T-T_c'),$$

where T_c' is a temperature very close to the critical; and this is only a more general statement of the relation we have just deduced.

As Eötvös has verified his relation experimentally for a large number of bodies, his result is a verification also of our general principles. The form of his relation also induces us to examine a little more closely an important consequence of the form of our infracritical equation, which, when multiplied by M with the pv term removed at low pressures, becomes

$$\frac{Ml}{2v}=R'M\left(1+\frac{\sqrt{T}}{B}\cdot\frac{k'-v}{v-\beta}\right)T.$$

Now R'M is constant, and Ml/v is proportional to $\alpha(Mv)^{\frac{2}{3}}$, if α and v are measured at $2T_c/3$, or any other constant fraction of the critical temperature, and under these circumstances $\alpha(Mv)^{\frac{2}{3}}$ has been shown to be proportional to T_c; hence if T is aT_c, a being a constant fraction, we get T_c proportional to

$$\left(1+\frac{\sqrt{T}}{B}\cdot\frac{k'-v}{v-\beta}\right)aT_c,$$

so that $1+\frac{\sqrt{\mathrm{T}}}{\mathrm{B}}\cdot\frac{k'-v}{v-\beta}$, measured at a constant fraction of the critical temperature, is approximately the same for all bodies, a result which our study of Van der Waals's generalization showed us to be approximately true. This shows that Eötvös's relation is rigorous only to the same extent as the constancy of this last expression is rigorous; as a matter of fact, excluding the alcohols and water, Eötvös finds the constant whose mean value is taken as ·227 to depart from this mean value by not more than 5 per cent. in any individual case. This brief discussion of Eötvös's relation has therefore furnished us with additional proof of the general accuracy of the approximations we have been forced to make in parts of our work.

One of the main difficulties in the way of pushing on with the many interesting inquiries opened up by these relations lies in the fact that we do not know the relation between the densities in the body and in the surface-layer of a liquid.

We have to replace our relation

$$\alpha \text{ varies as } \mathrm{A}\bar{\rho}^{\frac{5}{3}}m^{\frac{1}{3}}$$

by the less accurate one,

$$\alpha \text{ varies as } \mathrm{A}\rho^{\frac{5}{3}}m^{\frac{1}{3}}.$$

Multiplying by $(\mathrm{M}v)^{\frac{2}{3}}$ or $(\mathrm{M}/\rho)^{\frac{2}{3}}$, we get the $\alpha(\mathrm{M}v)^{\frac{2}{3}}$ of Eötvös proportional at all temperatures to $\mathrm{AM}\rho$; and as $d\{\alpha(\mathrm{M}v)^{\frac{2}{3}}\}d\mathrm{T}$ is constant, and has been shown by Eötvös to be constant almost right up to the critical temperature, and to be the same for all bodies, we ought, if our assumptions were rigorous, to have $\mathrm{AM}d\rho/d\mathrm{T}$ constant almost up to the critical temperature and the same for all bodies (whence another approximate method of finding A or l). Now $d\rho/d\mathrm{T}$ has been shown by Mendeléeff to be constant for many substances within ordinary temperature-ranges; but the constancy does not hold up to the critical temperature, and the ultimate meaning of the apparent contradiction between Eötvös's result and this is that, while for most purposes we may safely enough assume $\bar{\rho}$ proportional to ρ, we cannot so accurately proceed to the consequence $d\bar{\rho}^{\frac{5}{3}}/d\mathrm{T}$ proportional to $d\rho^{\frac{5}{3}}/d\mathrm{T}$; in fact, a change of temperature being accompanied by a change of stress in the surface-layer, the change of $\bar{\rho}$ with temperature is more complex than that of ρ. But within the range of temperature for which $d\rho/d\mathrm{T}$ is approximately constant, we have the important result that

$$\frac{d}{d\mathrm{T}}\,\mathrm{MR}'\left(1+\frac{\sqrt{\mathrm{T}}}{\mathrm{B}}\cdot\frac{k'-v}{v-\beta}\right)\mathrm{T}$$

is constant and is approximately the same for all bodies.

As I now consider the term differentiated to be not two thirds of the translatory kinetic energy of the gramme-molecule, but two thirds of the sum of the total kinetic energy and the chemic virial, I must replace the verbal statement of the last result as given in my paper (Phil. Mag. April 1889, p. 312) by the following :—The temperature-rate of variation of the sum of the total kinetic energy and the chemic virial of a molecule, measured at low constant pressure, is the same for all bodies (approximately).

10. *Tabulation of Values of the Virial Constant*, determined by four of the five methods described, and multiplied by the square of the molecular weight.—In the first place, I will give a comparison of the values of $\mathrm{M}^2 l$ for those bodies to which existing data allow the application of three or four of the previously described methods. The multiplication by M^2 is for future convenience. The values obtained by the second (liquid compression and expansion), third (latent heat), fourth (critical temperature), and fifth (capillarity) methods are entered in the columns marked 2, 3, 4, and 5. (The modified fourth method was used, namely, $\mathrm{M}l=800\,\mathrm{T}_c v_1$.)

The units are the megamegadyne (10^{12} dynes), grm., and cm.

The satisfactory agreement of these values, calculated for such diverse bodies from such diverse data, must be taken as the verification of the main principles so far unfolded—the chief of which as regards molecular force is that for most compound bodies the internal virial term of the characteristic equation is $l/2v$ below the volume k, and $l/(v+k)$ above that volume.

TABLE XXIV.

Substance.	2.	3.	4.	5.
CS_2	26·5	25·7	27·2	26·9
PCl_3	......	39·7	41·7	43·4
CCl_4	......	46·3	46·2	45·6
$CHCl_3$	33·0	38·2	36·1	36·8
C_2H_5Cl	22·3	28·3	26·5	
C_2H_5Br	......	32·1	31·5	29·0
C_5H_{12}	48·0	......	45·3	47·1
C_6H_{14}	57·0	......	58·5	59·3
C_6H_6	40·0	43·1	42·7	43·8
C_7H_8	59·0	55·8	56·2	56·4
$(CH_3)_2CO$	......	36·0	31·3	31·1
Methyl butyrate	65·0	60·4	55·6	56·1
Ethyl butyrate	84·0	74·6	69·5	71·3

We see, too, now how important for molecular dynamics is the detailed study of each of the constants k, B, and β in the characteristic equations ; but for the present we must refrain from entering on such a study, and must consider the comparison in the last table as closing for the present the general discussion of the characteristic equation.

Our immediate object is now to ascertain the law connecting the value of M^2l for a body with its chemical composition. On the hypothesis of the inverse fourth power (with the subsidiary one of molecular swarms),

$$m^2l = \pi m^2 A(4 \log 2L/a - 16/3) ;$$

so that, the bracketed expression being the same for all bodies, m^2l is proportional to m^2A, and the law of m^2l or M^2l will be the law of m^2A in the expression $3Am^2/r^4$ for the force between two molecules. In the Phil. Mag. for April 1889 I announced a law for the parameter A, calculated from Schiff's capillary data, which applied fairly well to a large number of organic compounds, but was affected with exceptions subversive of its generality. Applying now the more accurate method of calculation described as the fifth to all Schiff's data, we obtain ample material for generalization, which can be supplemented by values calculated by the other methods.

In the following Table the units are again the megamegadyne, grm., and cm. For brevity the radicals methyl, ethyl, and propyl, &c., will be denoted by the first two letters of their names, while the acid radicals—formic, propionic, &c.—will be denoted by Fot, Prt, and so on, so that PrPrt stands for propyl propionate. The numbers entered under the heading S will be explained when we are discussing the law of M^2l.

TABLE XXV.—Values of M^2l.

First Method.

	M^2l.	S.		M^2l.	S.
Et_2O	40·2	4·5	H_2	·22	·04
CO_2	7·1	1·07	N_2	1·23	·205
SO_2	15·0	2·05	O_2	1·16	·195
NH_3	8·5	1·25	CH_4	2·2	·35
N_2O	8·8	1·3	C_2H_4	6·5	1·0

Third Method.

	M^2l.	S.		M^2l.	S.
$SnCl_4$	61·3	6·1	IsoBuAct	76·2	7·1
$AsCl_3$	49·0	5·2	EtBut	76·9	7·15
BCl_3	23·5	2·95	PrPrt	77·3	7·2
$SiCl_4$	47·3	5·05	IsoAmFot	78·7	7·3
CH_3I	27·4	3·3	PrisoBut	89·1	8·0
EtI	40·0	4·5	EtVat	89·5	8·0
AmI($C_5H_{11}I$)	88·3	7·9	IsoBuPrt	92·5	8·15
AmBr	61·8	6·15	IsoAmAct	93·1	8·15
AmCl	48·7	5·15	PrBut	92·5	8·15
C_5H_{10}	35·0	4·0	IsoBuisoBut	105	8·9
$C_{10}H_{20}$	90·4	8·0	PrVat	106	8·9
$C_{10}H_{16}$	109·0	9·1	IsoBuBut	109	9·1
$C_2H_4Br_2$	49·4	5·2	IsoAmPrt	110	9·15
$Et_2C_2O_4$	109·0	9·1	IsoAmisoBut	125	9·95
Et_2O	44·1	4·8	IsoPrVat	126	10
EtFot	35·6	4·1	IsoAmBut	129	10·15
MeAct	36·0	4·1	IsoAmVat	147	11
EtAct	47·6	5·1	C_6H_6	43·1	4·7
MePrt	47·5	5·1	C_7H_8	55·8	5·7
PrFot	50·2	5·3	C_6H_5Et	69·5	6·65
MeButiso	60·4	6·04	C_8H_{10} (meta)	71·4	6·8
IsoBuFot	63·7	6·3	C_6H_5Pr	85·9	7·75
EtPrt	62·4	6·2	C_9H_{12}(mesitylene)	86·1	7·75
PrAct	62·8	6·2	C_9H_{12} (pseudo-cumene)	87·1	7·8
MeBut	61·8	6·15			
EtisoBut	74·6	7·0	$C_{10}H_{14}$ (cymene)	101	8·6
MeVat	74·1	7·0			

Table XXV. (*continued*).

Fourth Method.

	M^2l.	S.		M^2l.	S.
Cl_2	5·8	·9	PrCl	34·6	4·0
H_2S	10·5	1·5	NH_3	10·0	1·4
C_2H_2	9·8	1·4	NH_2Me	17·6	2·3
C_2H_6	14·7	2·0	$NHMe_2$	23·5	3·0
C_5H_{10}	44·1	4·8	NMe_3	31·6	3·7
$CHCl_3$	36·9	4·2	NH_2Et	21·2	2·7
CS_2	27·0	3·3	$NHEt_2$	41·4	4·55
C_2N_2	17·7	2·3	NEt_3	65·5	6·4
HCl	7·9	1·15	NH_2Pr	33·4	3·9
CH_3Cl	15·6	2·1	$NHPr_2$	67·6	6·5
EtCl	26·6	3·25			

Fourth Method modified (not using critical pressures).

	M^2l.	S.		M^2l.	S.
$(CH_2Cl)_2$	38·0	4·3	EtC_3H_5O	48·6	5·15
CH_3CHCl_2	37·7	4·3	HCl	7·3	1·1
C_3H_5Cl	34·0	4·0	MeCl	16·8	2·2
$CH_2(OCH_3)_2$ (methylal)	38·5	4·3	EtCl	26·4	3·2
			PrCl	32·6	3·8
$CH_3CH(OC_2H_5)_2$ (acetal)	68·7	6·6	NH_3	7·6	1·1
			NH_2Me	15·6	2·1
C_5H_{10}	41·8	4·6	$NHMe_2$	23·5	3·0
C_6H_{10}	51·2	5·4	NMe_3	31·0	3·7
C_8H_{18}	80·5	7·4	NH_2Et	24·4	3·05
C_8H_{16}	81·1	7·4	$NHEt_2$	43·5	4·75
Me_2O	20·3	2·6	NEt_3	66·4	6·5
MeEtO	29·9	3·6	NH_2Pr	31·0	3·7
EtPrO	52·4	5·45	$NHPr_2$	68·6	6·6

Fifth Method.

	M^2l.	S.		M^2l.	S.
EtFot	32·1	3·8	EtBut	72·3	6·85
MeAct	35·6	4·1	EtisoBut	71·3	6·8
PrFot	44·2	4·8	MeVat	69·7	6·7
EtAct	45·4	4·9	IsoAmAct	87·3	7·8
MePrt	44·8	4·85	IsoBuPrt	87·7	7·85
IsoBuFot	56·9	5·8	PrBut	89·3	7·95
PrAct	57·9	5·85	PrisoBut	87·7	7·85
EtPrt	58·5	5·9	EtisoVat	86·5	7·8
MeBut	58·2	5·9	IsoAmPrt	105	8·85
MeisoBut	56·1	5·75	IsoBuBut	106	8·9
IsoAmFot	71·6	6·8	IsoBuisoBut	103	8·75
IsoBuAct	70·6	6·75	PrisoVat	103	8·75
PrPrt	73·3	6·9			

Table XXV. (*continued*).

		M^2l.	S.
C_6H_{14}	Normal hexane	59·3	6·0
C_8H_{18}	Diisobutyl	90·6	8·0
$C_{10}H_{22}$	Diisoamyl	125·6	10·0
C_5H_{10}	Amylene	45·0	4·9
C_8H_{16}	Caprylene	91·8	8·1
$C_{10}H_{16}$	Terpene	107·4	9·0
C_6H_{10}	Diallyl	54·8	5·6
C_6H_6	Benzene	43·8	4·75
C_7H_8	Toluene	56·4	5·75
C_8H_{10}	Orthoxylene	69·7	6·65
C_8H_{10}	Metaxylene	69·7	6·65
C_8H_{10}	Paraxylene	70·3	6·7
C_8H_{10}	Ethylbenzene	70·6	6·7
C_9H_{12}	Normal propylbenzene	86·9	7·8
C_9H_{12}	Ethyltoluene	86·1	7·75
C_9H_{12}	Mesitylene	85·3	7·65
$C_{10}H_{14}$	Cymene	103·0	8·75
$CHCl_3$	Chloroform	36·9	4·2
CCl_4	Carbon tetrachloride	45·6	4·9
$(CH_2Cl)_2$	Ethylene chloride	44·4	4·8
CH_3CHCl_2	Ethidene chloride	38·1	4·3
PrCl	Propyl chloride	37·9	4·3
IsoBuCl	Isobutyl chloride	52·0	5·4
IsoAmCl	Isoamyl chloride	66·4	6·45
C_6H_5Cl	Chlorobenzene	57·7	5·85
C_7H_7Cl	Chlorotoluene	76·0	7·1
C_7H_7Cl	Benzyl chloride	82·2	7·5
$C_3H_6Cl_2$	Propylene chloride	56·8	5·8
C_2Cl_4		65·6	6·4
$C_2H_3Cl_3$	Trichlorethane	58·2	5·9
C_3H_5ClO	Epichlorhydrin	49·8	5·25
CCl_3COH	Chloral	50·9	5·35
CH_2ClCO_2Et	Ethyl chloracetate	63·2	6·25
$CHCl_2CO_2Et$	Ethyl dichloracetate	79·6	7·35
CCl_3CO_2Et	Ethyl trichloracetate	93·5	8·2
C_7H_5OCl	Benzoic chloride	79·6	7·35
$C_7H_6Cl_2$	Benzilidene chloride	89·4	8·0
EtBr	Ethyl bromide	28·9	3·5
PrBr	Propyl bromide	41·4	4·6
IsoPrBr	Isopropyl bromide	41·2	4·6
C_3H_5Br	Allyl bromide	39·9	4·45
IsoBuBr	Isobutyl bromide	53·3	5·5
IsoAmBr	Isoamyl bromide	66·2	6·45
C_6H_5Br	Bromobenzene	66·8	6·5
C_7H_7Br	Orthobromotoluene	81·5	7·45
$C_2H_4Br_2$	Ethylene bromide	51·2	5·35
$C_3H_6Br_2$	Propylene bromide	61·5	6·1
MeI	Methyl iodide	25·6	3·15
EtI	Ethyl iodide	38·1	4·3
PrI	Propyl iodide	50·9	5·3
IsoPrI	Isopropyl iodide	49·8	5·25
C_3H_5I	Allyl iodide	50·2	5·3
IsoBuI	Isobutyl iodide	62·4	6·2
IsoAmI	Isoamyl iodide	80·0	7·35

Table XXV. (*continued*).

		M²*l*.	S.
C_6H_5I	Iodobenzene	80·4	7·35
NH_2Pr	Propyl amine	37·8	4·25
$NH_2C_3H_5$	Allyl amine	36·6	4·2
NH_2isoBu	Isobutyl amine	50·6	5·3
NH_2isoAm	Isoamyl amine	64·4	6·3
$NHEt_2$	Diethyl amine	50·1	5·5
NEt_3	Triethyl amine	75·2	7·05
$NH_2C_6H_5$	Aniline	65·0	6·35
C_5H_5N	Pyridine	59·7	6·0
C_9H_7N	Piperidine	100·3	8·6
C_9H_7N	Chinoline	108·6	9·05
$MeNO_2$	Nitromethane	22·6	2·75
CCl_3NO_2	Chloropicrin	57·6	5·85
$EtNO_3$	Ethyl nitrate	37·2	4·2
IsoAmNO_3	Isoamyl nitrate	76·9	7·2
C_3H_7CN	Isobutylnitrile	39·2	4·4
C_4H_9CN	Capronitrile	50·5	5·3
C_6H_5CN	Benzonitrile	65·8	6·4
CS_2	Carbon disulphide	26·9	3·3
$CSNC_3H_5$	Allyl sulphocarbimid	54·0	5·55
$CNSCH_3$	Methyl sulphocyanate	33·7	3·9
$CNSC_2H_5$	Ethyl sulphocyanate	44·1	4·8
Et_2S	Ethyl sulphide	55·0	5·6
PCl_3		43·4	4·75
$POCl$		25·0	3·1
POC_2H_5Cl	Ethoxyphosph. chloride	62·8	6·2
$PSCl$		35·5	4·1
$(CH_3)_2CO$	Acetone	31·1	3·7
$C_6H_{12}O_3$	Paraldehyde	85·7	7·7
$CH_3CH(OCH_3)_2$	Dimethylacetal	53·3	5·5
$CH_3CH(OC_2H_5)_2$	Diethylacetal	87·7	7·85
Et_2O	Ethyl oxide	43·8	4·75
$(CH_3CO)_2O$	Acetic anhydride	51·9	5·4
$CH_3CO_2C_3H_5$	Allyl acetate	54·1	5·55
MeisoAmO	Methylisoamyl oxide	66·8	6·5
$Et_2C_2O_4$	Ethyl oxalate	91·0	8·05
$C_6H_5CO_2CH_3$	Methyl benzoate	91·4	8·1
$C_6H_5CO_2C_2H_5$	Ethyl benzoate	109·9	9·15
$CH_3(CO)_2CH_2OEt$	Acetacetic ether	81·5	7·45
$C_6H\ OCH_3$	Anisol	70·0	6·7
$C_6H_5OC_2H_5$	Phenethol	80·7	7·4
$C_7H_7OCH_3$	Methyl paracresolate	84·1	7·6
$C_6H_4(OCH_3)_2$	Dimethyl resorcin	96·2	8·35
C_4H_2OHCOH	Furfurol	51·2	5·3
C_4H_9COH	Valeraldehyde	48·0	5·1
$C_9H_{11}COH$	Cuminol	113·0	9·3
$C_{10}H_{14}O$	Carvol	117·8	9·6
$(CH_3)_3CCOCH_3$	Pinakoline	63·5	6·3

The following are the chief sources of the data from which the above table has been constructed:—Latent Heats from Berthelot (*Chimie Mécanique*) and R. Schiff (*Ann. der Chem.* ccxxxiv.). Critical Temperatures from Sajontschewski (Wied. *Beibl.* iii.), Pawlewski (*Ber. deut. chem. Ges.* xv., xvi.), Nadejdine (Wied. *Beibl.* vii.), and Dewar (Phil. Mag. xviii.). Surface-tensions from R. Schiff (*Ann. der Chem.* ccxxiii.; Wied. *Beibl.* ix.).

To make clear the two simple laws that rule these tabulated values, it will be advisable to confine our attention at first only to those values obtained by the fifth method, using those by other methods only to fill gaps. It will also be well to consider first a single chemical family, such as the paraffins, for which we have the following values :—

CH_4.	C_2H_6.	C_6H_{14}.	C_8H_{18}.	$C_{10}H_{22}$.
2·2	14·7	59·3	90·6	125·6

These show that there is not a constant difference in the value of M^2l corresponding to the difference in the number of CH_2 groups contained. We can amplify this list of paraffins by using the material furnished by Bartoli and Stracciati (*Ann. de Chim. et de Phys.* sér. 6, t. vii.), who have determined the more important physical constants for all the paraffins from C_5H_{12} up to $C_{16}H_{34}$. Their values of the capillary constants were found at about 11 degrees in every case. To obtain those at two thirds of the critical temperature, we have to use the values of the critical temperatures calculated by them from Thorpe and Rücker's convenient empirical relation (Journ. Chem. Soc. xlv.),

$$\rho/(2T_c - T) = \text{constant},$$

ρ being density at T.

The following are the values of M^2l thus obtained :—

C_5H_{12}.	C_6H_{14}.	C_7H_{16}.	C_7H_{16}.	C_8H_{18}.	C_9H_{20}.	$C_{10}H_{22}$.
47·2	58·1	77·1	79·6	91·2	110	127

$C_{11}H_{24}$.	$C_{12}H_{26}$.	$C_{13}H_{28}$.	$C_{14}H_{30}$.	$C_{15}H_{32}$.	$C_{16}H_{34}$.
147	171	192	215	230	256

Considering the assumptions involved in the calculation of these values and the difficulty of obtaining the paraffins pure, the agreement for C_6H_{14}, C_8H_{18}, and $C_{10}H_{22}$ with Schiff's numbers is excellent; but the higher we go in the series the larger is the temperature-interval for which we have to extrapolate, and the more uncertain do the values become. However, they are useful as giving a general idea of the course of M^2l in an extended series.

On plotting these numbers as ordinates with abscissæ representing the number of CH_2 groups in the molecule, a curve was constructed which proved to be the parabola

$$M^2l = 6S + {\cdot}66S^2,$$

where S is the number of CH_2 groups. It is only necessary then to determine on this curve the abscissa corresponding to the value of M^2l for any substance to obtain the number of CH_2 groups to which its molecule is equivalent as regards molecular force.

11. *Definition of the Dynic Equivalent of a substance, and determination of its value for several Radicals.*—In the manner just described were obtained the numbers entered in Table XXV. under the heading S, which I propose to call the Dynic Equivalents of the substances. The dynic equivalent of a molecule is the number of CH_2 groups in the molecule of the normal paraffin that exerts the same molecular force as it.

The table shows that the atom of an element contributes the same amount towards the dynic equivalent of all molecules in which it occurs (except in the case of the simpler typical compounds) : thus, for example, consider the iodides from methyl to amyl iodide, and notice that each CH_2 group has the same value unity as in the paraffins, and that the iodine atom is equivalent to about $2{\cdot}3CH_2$ in every case. The same law holds throughout the table ; so that each elementary atom or radical has its own dynic equivalent, which can be easily determined.

In the first place, it appears that the extra H_2 in the paraffins C_nH_{2n+2} can be neglected in a first approximation, because C_5H_{10} has practically the same dynic equivalent as C_5H_{12}, and C_8H_{16} the same as C_8H_{18}. It may be that the double binding in the unsaturated compounds compensates for the H_2 of the saturated, but I think that the simpler idea for the present is that the two terminal H atoms in a paraffin chain have a dynic equivalent so small that we may neglect it, or more generally the middle H in a CH_3 group is negligible in a first approximation. If there is really such a difference between the middle H and the two others in CH_3, we ought to find the dynic equivalents for the iso-compounds smaller than for the normal ; and the table shows that the isobutyrates have equivalents smaller by ·1 than the butyrates, while the isobutyl salts of the fatty acids have dynic equivalents nearly all less than 1 greater than the propyl salts. But this is rather a matter for the chemist to work out in detail ; it suffices to indicate the idea here, and to point out that it is in harmony with the lowering of boiling-points among isomers with increasing number of CH_3 groups. Accordingly, as a matter of detail, in the estimation of the dynic equivalents of the elements, it was assumed that the equivalent of C_nH_{2n+2}, C_nH_{2n+1}, and C_nH_{2n} is in each case n when normal and $n - {\cdot}1p$ when the molecule departs from normality by p CH_3 groups. (In constructing my curve for dynic equivalents, for the sake of simplicity at first, I ignored the fact that two of the paraffins in Schiff's data are iso-compounds.)

From the tabulated values for the alkyl salts of the fatty acids, we get the following mean values for the dynic equivalent of $CO''O'$:—in the formiates 1·85, acetates 1·92, propionates 1·91, butyrates 1·92, isobutyrates 1·91, and isovalerates 1·83, the mean for all being 1·9.

From the oxides (ethers) and other compounds containing single-bound O, we find for its mean value ·6, although in the ethers containing the benzene nucleus it comes out ·8, while in the benzoates $CO''O'$ is about 2·5; so that the junction of the benzene nucleus with other groups seems to be accompanied with an increase in its value: witness also the bromide, iodide, and amine of C_6H_5, equal to the bromide, iodide, and amine of C_5H_{11}, although C_6H_6 is less than C_5H_{12} by ·25. This slight variation of the value of C_6H_6 is the only anomaly amongst the numbers of Table XXV. excepting the simple typical compounds. The values ·6 for O' and 1·9 for $C''O'$ are in harmony with the results for all the other compounds containing oxygen.

From the values for the benzene series of hydrocarbons we can get a value for H_2 in CH_2, and also some light on the important question of the structure of the benzene nucleus. Thomsen has been led by his thermochemical investigations to the conclusion that in benzene each carbon atom is bound to three other carbon atoms by a single bond to each (the "bond" phraseology is used merely for brevity, and not as expressing definite statical or dynamical facts). If we accept this conclusion, then the dynic equivalent of C_6H_{12} minus that for C_6H_6 is equal to the dynic equivalent of $3H_2$; similarly, from the other members of the benzene series we get the values for $3H_2$, the mean result being 1·29 or ·43 for H_2.

The accepted structure for C_6H_{10} diallyl is $(CH_2CHCH_2)_2$, or four whole CH_2 groups and two CH_2 groups with H removed from each; and as the dynic equivalent of C_6H_{10} is 5·6, we find that of H_2 to be ·4. Again, recent investigations on the terpenes (Wallach, *Ann. der Chem.* ccxxv., ccxxvii., ccxxx.; Brühl, ccxxxv.) show that the two ordinary forms can have their formulæ written $CH_3C_3H_7(CH_2)_2(CH)_2C_2$; that is to say, they have 6 hydrogen atoms cut out of CH_2 groups, and as the dynic equivalent of $C_{10}H_{16}$ is 9 we have that of $3H_2$ as 1·0 or that of H_2 ·33. Since the value ·43 is derived from 10 accordant members of the benzene series, we will take it as the value of the dynic equivalent of H_2 in CH_2.

By similar but simpler reasoning the dynic equivalents of Cl, Br, I, and other radicals are easily found, and the following is a list of mean values:—

TABLE XXVI.—Dynic Equivalents.

CH_2.	H.	C.	$C''O'$.	O'.	NH_2.	CN.
1·0	·215	·57	1·9	·6	1·23	1·35

NO_3.	CNS.	S'.	Cl.	Br.	I.
2·2	2·85	1·6	1·3	1·6	2·3

To illustrate the applicability of these values, I furnish a comparison of the values calculated by means of them for twenty substances with the values tabulated in Table XXV.

TABLE XXVII.

Comparison of calculated and tabulated Dynic Equivalents.

	Calc.	Tab.		Calc.	Tab.
C_6H_6	4·8	4·75	C_4H_9CN............	5·35	5·3
$C_{10}H_{14}$	8·8	8·75	$C_4H_{10}S$	5·6	5·6
C_3H_7Cl	4·3	4·3	C_2H_5CNS	4·85	4·8
$C_5H_{11}Cl$	6·3	6·45	$(CH_3)_2O$............	2·6	2·6
C_2H_5Br	3·6	3·5	$CH_3C_5H_{11}O$	6·6	6·5
$C_5H_{11}Br$............	6·6	6·45	$C_6H_{12}O_3$............	7·8	7·7
C_2H_5I	4·3	4·3	$(CH_3CO)_2O$	5·2	5·4
$C_5H_{11}I$	7·3	7·35	C_4H_9COH	5·3	5·1
$NH_2C_3H_7$	4·23	4·25	$C_4H_8O_2$	4·9	4·9
$NH_2C_5H_{11}$	6·23	6·3	$C_7H_{14}O_2$............	7·9	7·9

If we now look at the dynic equivalents of the uncombined elements given in the first part of Table XXV., we may notice that they are remarkably small compared to the values in the combined state; thus, that of H_2 is ·04, of N_2 ·205, of O_2 ·195, and that of CH_4 is small too, ·35, instead of 1 as it should be, seeing that CH_4 is the first of the paraffin series. Other typical compounds have small values: CO_2 has 1·05, while $CO''O'$ in more elaborate compounds has a value 1·9, C_2H_4 has 1, while C_2H_6 has 2; and so on. The same fact has been noticed in connexion with the molecular refraction of some of the typical compounds and some of their immediate derivatives, and it will yet prove a most important one in chemical dynamics. But meanwhile it is of greater importance for present purposes to notice that the dynic equivalents given for various radicals in Table XXVI. are closely proportional to their molecular refractions.

12. *Close parallelism between Dynic Equivalents and Molecular Refractions.*—As is well known, there are two methods according to which the molecular refraction is estimated, the first by means of Gladstone's expression, $(n-1)M/\rho$, where n is index of refraction; the other by means of Lorenz's, $(n^2-1)M/(n^2+2)\rho$.

In a brief paper (Phil. Mag. Feb. 1889) I showed that the experimental evidence taken as a whole is in favour of the Gladstone expression, for which also a very simple theoretical proof can be given; and, further, it was shown that it is best to measure $(n-1)M/\rho$ if possible in the gaseous state. But as comparatively few measurements have been made on bodies in the vapour state I suggested that, as the Lorenz expression had been empirically proved to give more nearly the same value in the liquid and vapour states of a body, its value as determined in the liquid state and multiplied by 3/2 could be taken as giving the value of $(n-1)M/\rho$ in the vapour state. The result of the theoretical argument was that, if M/ρ is taken to measure the molecular domain u, and if U is the volume occupied by the molecule in the same units, and N is the index of refraction for the matter of the molecule, then

$$(n-1)u = (N-1)U.$$

Landolt, Brühl, and others have determined the values of the atomic refraction for several elements (*Ann. der Chem.* ccxiii. p. 235), and by means of these and Masini's data for sulphur (Wied. *Beibl.* vii.) and Gladstone's latest determinations (Journ. Chem. Soc. 1884), I have obtained the values of the refraction-equivalents of the preceding radicals in terms of that for CH_2 as unity. Mascart has given (*Compt. Rend.* lxxxvi.) values of the refraction of a number of substances in the vapour state, from which, for the sake of comparison, I have calculated the refraction-equivalents for as many radicals as possible.

The following Table contains in the second column the dynic equivalent, in the third the refraction-equivalent calculated according to the Lorenz expression, in the fourth the refraction-equivalent calculated according to the Gladstone expression from Mascart's data for vapours, and in the fifth that calculated by Gladstone from liquid data. The value for CH_2 in every case is 1.

TABLE XXVIII.

Comparison of Dynic and Refraction Equivalents.

1.	2.	3.	4.	5.
CH_2	1·0	1·0	1·0	1·0
H	·215	·23	·19	·17
C	·57	·54	·62	·66
CO″O′	1·9	1·4	1·5	1·5
O′	·6	·35	·4	·37
NH_2	1·23	1·12	......	1·01
CN..................	1·35	1·18	1·3	1·2
NO_3	2·2	2·2	......	1·9
CNS	2·85	......	......	3·0
S′	1·6	1·7	......	1·9
Cl	1·3	1·3	1·5	1·3
Br	1·6	2·0	1·7	2·0
I.....................	2·3	3·1	2·7	3·2

This table brings out the remarkable fact that the parallelism between the dynic and refraction-equivalents is so close as almost to amount to proportionality. I shall not discuss the meaning of this relation until I have shown how to obtain the dynic equivalents for the elements usually occurring in inorganic compounds, and established the same relation for them also.

Meanwhile it will be useful to compare the dynic and refraction equivalents of the uncombined elements and of those simpler typical compounds to which we have said the summative law does not apply as regards dynic equivalents and does not accurately apply as regards refraction-equivalents.

TABLE XXIX.

Ratio of Dynic to Refraction Equivalents, each measured in terms of that for CH_2 as unity.

H_2.	O_2.	N_2.	Cl_2.	CH_4.	C_2H_4.	H_2S.	C_2H_2.	$CHCl_3$.	CS_2.
·09	·21	·22	·38	·32	·45	·79	·75	·94	·72
C_2N_2.	HCl.	CH_3Cl.	NH_3.	CO_2.	SO_2.	N_2O.	PCl_3.	CCl_4.	$(CH_3)_2O$.
·90	·83	·79	1·06	·76	·96	·83	·88	·89	·96

We see that the ratio is small for the uncombined elements and CH_4 and C_2H_4, a result of the important fact that while the refraction-equivalent of a non-metallic element is almost the same in the uncombined state as in the combined, the dynic equivalent is much smaller in the uncombined state of an element. The meaning of this fact will be discussed later on ; connected with it is the fact that the ratio for most of the typical compounds in the last table is notably less than unity. A molecule has to reach a certain degree of complexity before the summative law holds as regards its dynic equivalent ; the same may be said about its heat of formation. Further comment on the connexion between dynic equivalent and heat of formation must be deferred for a little.

We can now give a formal enunciation of the law of the virial constant :—If the molecule of an organic compound is of a degree of complexity higher than that of the ordinary typical compounds, then the virial constant for one gramme of the compound is given in terms of the megamegadyne, gramme, and centimetre as units by the relation

$$M^2 l = 6\,S + \cdot 66\,S^2,$$

where M is the usual molecular weight of the compound, and S is the sum of the dynic equivalents of the atoms in its molecule (measured in terms of that for CH_2 as unity).

According to the law of force $3\,Am^2/r^4$, Am^2 is proportional to $M^2 l$ and therefore follows the same law.

13. *Return to the Discontinuity during Liquefaction of Compounds and proof that it is due to the pairing of Molecules.*—The interpretation of the form of the internal virial expression above the volume k is of the highest importance in the theory of molecular force, and can now be attempted in the light of the law for $M^2 l$. If the molecules of a substance do pair to produce an actual chemical polymer of it, then its molecular mass changes from M to 2M, while S the dynic equivalent changes to 2S, and consequently

$$l \text{ or } (6\,S + \cdot 66\,S^2)/M^2$$

changes to a value given by

$$l' = (6S/2 + \cdot 66\,S^2)/M^2.$$

When S is small we see that the pairing of molecules to produce a polymer causes the virial constant (for one gramme) to diminish to nearly the half of its value for free molecules; when S is larger this statement becomes less exact. In the case of CO_2, for example, if from Table XXV. we take S as 1·05, then

$$l = \cdot 00367 \quad \text{and} \quad l' = \cdot 00206,$$

but in the case of ethyl oxide, with $S = 4\cdot 5$,

$$l = \cdot 00738 \quad \text{and} \quad l' = \cdot 00491.$$

In both cases we see that the pairing of the molecules to form a new chemical compound or polymer is attended with a reduction of the virial constant towards one half, but not exactly to one half of the original value. Now the data given in Tables I. and III. and the form of virial term taken to represent them, $l/(v+k)$, along with the fact that below volume k the form is $l/2v$, mean that in the limiting gaseous state the virial term is practically l/v, just as below k it is $l/2v$; hence we must regard the pairing of the molecules to be such as to cause the virial constant to become one half of its higher limit—in other words, the pairing must be different from polymerization. We are therefore led to differentiate the chemical and physical pairing of molecules by the statement that while chemical pairing alters the virial constant in the ratio

$$(6\,\mathrm{S}+\cdot 66\,\mathrm{S}^2)/(12\,\mathrm{S}+2\cdot 64\,\mathrm{S}^2) \quad \text{or} \quad (1+\cdot 11\,\mathrm{S})/(2+\cdot 44\,\mathrm{S}),$$

physical pairing alters it in the ratio

$$6\,\mathrm{S}/12\,\mathrm{S} \quad \text{or} \quad 1/2.$$

The term $\cdot 66\,\mathrm{S}^2$ would thus appear to have a certain chemical significance.

And now as to the form $l/(v+k)$ connecting the two extreme cases. We can explain it in the following manner:— It will be shown when we come to treat of solutions that if a salt having a parameter of molecular force $3\,\mathrm{A}$ (proportional to its virial constant l) is dissolved in a solvent with parameter $3\,\mathrm{W}$ so that there are n molecules of salt to one of solvent, then the solution behaves as if it consisted of molecules having a parameter $3\,\mathrm{X}$ given by the singular relation

$$\mathrm{X}^{-1} = (\mathrm{W}^{-1}+n\mathrm{A}^{-1})/(1+n).$$

Now, in a gas being compressed towards the volume k, let us assume that there are a number of pairs of molecules proportional to k and a number of single molecules proportional to $v-k$, and that the same relation applies to the mixture of paired and single molecules as to the solvent and salt in a solution, then, replacing W by l and A by $l/2$ and n by $k/(v-k)$, we get for the reciprocal of the virial constant of the mixture

$$\mathrm{X}^{-1}=\left(\frac{1}{l}+\frac{k}{v-k}\cdot\frac{2}{l}\right)\Big/\left(1+\frac{k}{v-k}\right)=\frac{1}{l}\cdot\frac{v+k}{v};$$

$$\therefore\ \mathrm{X}=lv/(v+k),$$

and therefore the virial term is $l/(v+k)$.

This would be no demonstration of the pairing of molecules in compounds, if we did not already have it proved in the case of the elements that the virial term varies inversely as the volume at all volumes. Remembering this we can accept our form $l/(v+k)$ as indicating the existence of a mixture of paired and single molecules, the number of pairs at volume v being to the number of single molecules as k to $v-k$.

The form of the energy term,

$$\mathrm{RT}\left(1+\frac{2k}{v+k}\right),$$

must also be partly determined by this existence of pairs, but it would be foreign to our immediate subject to attempt to investigate it.

14. *Brief Discussion of the Constitution of the Alcohols as Liquids.*—To learn a little more on the subject of pairing, it will be of some profit to consider briefly here the alcohols and water, which so far have been left aside, after having been proved in Table XII. to follow in the supracritical region a different law from the usual one. But also in the liquid state the alcohols and water, while conforming to the general liquid laws in many respects, are still exceptional in others. Thus Eötvös (Wied. *Ann.* xxvii.) has shown that the alcohols will conform to his generalization if at the lower range of temperature, from 20° C. to 170° C., the molecules be considered complex (double relatively to ordinary liquids), and water also conforms if from 100° upwards its molecules be considered double; the molecular lowering of the freezing-point of water produced by the solution of bodies in it as measured by Raoult and compared with the molecular lowering for other liquids proves that relatively to these the molecule of water is double. Taking all the facts into consideration, it seems to me that the alcohols and water may be assumed, in the liquid state, to have the pairs of molecules again paired, the second pairing, however, not being of so intimate a nature as the first, consisting of a mere approximation of the first pairs without any change in the values of A or of l.

According to this assumption we should expect the behaviour of the liquid alcohols to be represented by a form similar to our infracritical equation, and we will assume

$$pv = \mathrm{R}''\mathrm{T}\left(1 + \frac{\sqrt{\mathrm{T}}}{\mathrm{B}} \cdot \frac{k'-v}{v-\beta}\right) - \frac{l}{2v'}$$

where R″ is about 2R.

It was on the infracritical equation that our second method of finding $\mathrm{M}^2 l$ was founded, giving the relation

$$l = \frac{4}{3}\left(v_0 \frac{a}{\mu} + \frac{25}{26}\mathrm{R}\right) v\mathrm{T},$$

which can be applied to Amagat's data (*Ann. de Chim. et de Phys.* sér. 5, t. xi.), and Pagliani and Palazzo's (Wied. *Beibl.* ix.) for the alcohols.

Again, our assumption enables us to apply the fifth or capillarity method of finding $\mathrm{M}^2 l$ to the alcohols, if only we remember that the molecular domain becomes twice as large and its radius $2^{\frac{1}{3}}$ as large as it would be with only one pairing; hence the equation for the fifth method becomes for the alcohols

$$l = c\,\alpha\, v^{\frac{4}{3}}/(2m)^{\frac{1}{3}}.$$

Schiff's data are available.

As the latent heat and critical temperature are largely dependent on the supracritical equation, we ought not to expect the formulæ for $\mathrm{M}^2 l$ furnished by the third or latent-heat method and by the fourth or critical-temperature method to apply to the alcohols. But as the relation

$$\mathrm{M}\lambda = 19{\cdot}4\mathrm{T}_b,$$

or, empirically,

$$\mathrm{M}\lambda = 21\,\mathrm{T}_b,$$

which was deduced in the discussion of the third method, does apply approximately to the alcohols, the constant being 26, we may as well, for purposes of comparison, see what the formulæ of the third and fourth methods give in the case of the alcohols.

TABLE XXX.—M^2l for the Alcohols.

Method	Second.	Third.	Fourth.	Fifth.
Methyl	8	22 (11)	17 (9)	9·4
Ethyl...............	14	38 (19)	25 (17)	17·6
Propyl	24	......	34 (26)	27·8
Isopropyl	24	......	33 (25)	27·6
Isobutyl............	34	......	46 (38)	37·8
Isoamyl............	47	82 (41)	57 (49)	48·1
Cetyl	......	326 (163)		
Water	......	11 (5·5)	9	6

The second and fifth methods give results in substantial agreement with one another; the third gives numbers which, when halved as in the brackets, come into agreement with the others; while if 8 be subtracted from the numbers furnished by the fourth method as in the brackets, the resulting numbers are in very close agreement with those given by the fifth method. That the two discrepant columns should be capable of harmonization with the two others by the simple operations of halving and of subtracting a constant is significant of some really simple principle on which the abnormality of the alcohols in the supracritical region depends.

But too much importance must not be attached to the general agreement among the numbers, as they are all founded on tentative assumptions ; they simply prove that our idea of a second loose pairing of molecular pairs in the liquid alcohols is not discordant with facts.

The following are the dynic equivalents corresponding to the values of M^2l found by the fifth method compared with those calculated from the dynic equivalents of the elements and placed in the third row.

Alcohols	Methyl.	Ethyl.	Propyl.	Butyl(iso).	Amyl(iso).
Dynic equiv. found	1·35	2·35	3·4	4·3	5·1
„ „ calcul.	1·8	2·8	3·8	4·7	5·7

For water the dynic equivalent found is ·91, while 1·03 is the calculated value. The values found for the alcohols are about ·5 smaller than those calculated. If from the capillarity data we had calculated the values of M^2l and the dynic equivalent as if the alcohols were not exceptional in any way, we should have got:—

Alcohols	Methyl.	Ethyl.	Propyl.	Butyl.	Amyl.
M^2l	11·8	22·2	35	47·6	60·6
Dynic equivalent	1·65	2·8	3·9	5·1	6·1

These values of the dynic equivalents agree well with the calculated ones above, but then the values of M^2l being all increased in the ratio of $2^{\frac{1}{3}}$ to 1 are no longer in agreement with the values got by the second method.

Accordingly we see that the alcohols will require an exhaustive study for themselves, if the interesting features of their molecular structure are to be thoroughly made out. I have merely sketched lines on which their abnormality may be hopefully investigated. Those bodies, such as nitric peroxide, studied by the brothers Natanson, and acetic acid, studied by Ramsay and Young, which have been proved to have double molecules split up both by the action of heat and reduction of pressure have not been touched on in this paper; they also would require a special investigation in which our characteristic equations would lend an assistance much required.

15. *Methods of finding the Virial Constant for Inorganic Compounds, including a Theory of the Capillarity and Compressibility of Solutions.*—So far I have secured only two methods of finding the virial constants and dynic equivalents of inorganic compounds from existing data, and only one of these is practically useful, namely the first, in which the surface-tensions of solutions are the source of the values; the second is based on the compressibility of solutions. Using our expression for surface-tension in terms of molecular force,

$$\alpha = \pi\rho^2 Ae/(2 + \sqrt{2}),$$

in the form

$$\alpha = A\rho^{\frac{4}{3}} m^{\frac{1}{3}}/c,$$

where c is a constant, I was led to imagine that it could be adapted to the case of a solution by means of the following suppositions:—First, that in a solution containing n molecules of dissolved substance of molecular mass p to one of solvent of molecular mass w, the solution may be assumed to be a substance of molecular mass

$$m = (w+np)/(1+n)\,;$$

second, that if 3W is the parameter of molecular force for the solvent and 3A for the dissolved substance, then the parameter 3X for the solution is connected with W and A by the relation

$$X^{-1} = (W^{-1}+nA^{-1})/(1+n).$$

This seems highly arbitrary, but will be completely verified by the results to which it leads. I could make no progress in the handling of solutions until, in the course of some work on the elasticity of alloys, I discovered a relation similar to the above to hold, and this proved to be the immediate clue to the treatment of solutions.

Let α_w be the surface-tension of water, then we have the following equations giving A :—

$$\alpha = X\rho^{\frac{4}{3}}\left(\frac{w+np}{1+n}\right)^{\frac{1}{3}}\Big/c, \qquad \alpha_w = Ww^{\frac{1}{3}}/c,$$

$$A^{-1} = X^{-1}+(X^{-1}-W^{-1})/n.$$

These equations ought to give the same values of A^{-1} whatever the strength of the solution may be ; and herein lies a first test of the truth of the principles involved.

The following values of cA^{-1} for NaCl are calculated from Volkmann's data (Wied. *Ann.* xvii.) for its solution in water at 20° ; w is taken as 18, although we consider the water molecule to be complex, but this does not affect the purely relative comparison being made :—

n . . .	·105	·084	·052	·035	·017
cA^{-1} . .	1·34	1·38	1·47	1·46	1·42

Considering that the solutions range from saturation down to considerable dilution, the approach to constancy is satisfactory; but it will be noticed that, on the whole, there is a tendency for the value of cA^{-1} to increase with diminishing concentration, and this same phenomenon is to be seen in the case of almost all Volkmann's solutions, most pronouncedly in that of $CaCl_2$:—

n . . .	·091	·068	·041	·021	·011
cA^{-1} . .	2·41	2·53	2·77	2·95	3·12

This case shows us that there is a certain amount of incompleteness in our theory of the capillarity of solutions, as indeed we ought to be surprised if there were not, when we try to apply our arbitrary definition of the molecular mass of a solution to one which contains 56 parts by weight of $CaCl_2$ to 100 of H_2O as the solution for which $n=\cdot091$ does, and also when we assume that the concentration in the surface-layer is the same as in the body-fluid at all strengths up to saturation. If our object were an exhaustive representation of the connexion between the surface-tension of a solution and its concentration, it would be easy to introduce a slight empirical alteration into the above equations to make them exhaustive. For instance, we might imagine that the effective value of W in a solution experiences a small change proportional to the concentration; but the equations as they stand will prove to be sufficient for our purpose if, in comparing solutions of different substances, we calculate cA^{-1} for the same value of n throughout.

In all subsequent calculations $n=18/1000$. The experimental data for surface-tensions of solutions are abundant, the chief that I know of and have used being those of Valson (*Compt. Rend.* lxx., lxxiv.), Volkmann (Wied. *Ann.* xvii.), Röntgen and Schneider (Wied. *Ann.* xxix.), and Traube (*Journ. für Chem.* cxxxix.).

The following Table contains the values of cA^{-1} for a certain number of compounds, the surface-tension being measured in grammes weight per linear metre and the half molecules of the salts of the bibasic acids being regarded as molecules.

TABLE XXXI.—Values of cA^{-1}.

	I.	Br.	Cl.	NO_3.	OH.	$\frac{1}{2}SO_4$.	$\frac{1}{2}CO_3$.
Li	4·15	2·46	·83	1·61	·83	1·61	
NH_4	3·99	2·21	·70	1·54	...	1·45	
Na	4·74	3·06	1·45	2·30	1·37	2·30	1·95
K	4·99	3·31	1·78	2·55	1·71	2·72	2·21

The study of this table brings out the fact that the differences of the numbers in any two rows or in any two columns are constant: thus the differences for the iodide and chloride of the four bases are in order 3·32, 3·29, 3·29, and 3·21, while the differences of cA^{-1} for the Na and Li salts of the monobasic acids are in order ·59, ·60, ·62, ·69, and ·54. Accordingly each atom contributes a certain definite part to the value of cA^{-1} for the molecule in which it occurs, and that part is independent of the other atoms in the molecule. I shall call this part the parameter-reciprocal modulus of the atom ; we have not at present sufficient data to get its absolute value in any case, but if we make Li our standard positive radical, and Cl the standard negative, we can calculate the average values of the difference between the parameter-reciprocal modulus of a radical and its standard,—thus in the case of iodine this difference is 3·28, and in the case of Na ·61, and so on.

Before tabulating these mean values I will give the values calculated for the salts of some other metals with the values for the same salts of Li subtracted.

TABLE XXXII.

	I.	Br.	Cl.	NO$_3$.	½SO$_4$.
½Mg – Li.........	...	...	·39	·35	·53
½Ca – Li.........	·44	·73	·67	·55	...
½Sr – Li.........	...	...	1·67	1·71	...
½Ba – Li.........	...	...	2·49	2·52	...
½Zn – Li.........	...	...	1·35	...	1·33
½Cd – Li.........	2·24	2·42	2·15	...	2·29
½Mn – Li.........	...	...	1·17	·93	1·22

To these we may add the following values, obtained from the sulphates ½Fe – Li 1·27, ½Ni – Li 1·19, ½Co – Li 1·15, ½Cu – Li 1·49, ⅓Al – Li ·6, ⅓Fe$_{\text{ic}}$ – Li ·5, and ⅓Cr$_{\text{ic}}$ – Li 1·0 ; and the two following from the nitrates Ag – Li 3·91, and ½Pb – Li 4·51.

The following Table contains the values of the parameter-reciprocal moduluses of the different metals minus that for Li and of the negative radicals minus that for Cl.

TABLE XXXIII.

Mean Values of Parameter-reciprocal Modulus for the Metals with that for Li subtracted.

Na.	K.	NH.	½Mg.	½Ca.	½Sr.	½Ba.	½Zn.	½Cd.	½Mn.
·61	·90	– ·15	·42	·65	1·7	2·5	1·33	2·3	1·2

½Fe$_{\text{ous}}$.	½Ni.	½Co.	½Cu.	Ag.	½Pb.	⅓Al.	⅓Fe$_{\text{ic}}$.	⅓Cr$_{\text{ic}}$.
1·27	1·2	1·15	1·5	3·9	4·5	·6	·5	1·0

The same for negative radicals with that for Cl subtracted:—

I.	Br.	NO_3.	OH.	$\frac{1}{2}SO_4$.	$\frac{1}{2}CO_3$.
3·28	1·56	·81	·04	·86	·46

It may be worth mentioning that these differences show a pretty close parallelism to the corresponding differences of the atomic refraction given by Gladstone (Phil. Trans. 1870), but not close enough to be worth dwelling on.

In the case of the organic bodies studied there was nothing analogous to this singular property possessed by these inorganic compounds of having the reciprocal of the parameter A of molecular force separable into a constant and definite parts contributed by each constituent of the molecule. Thus we have characteristic of inorganic bodies in solution another of those properties called by Valson modular, who discovered that the density, capillary elevation, and refraction of normal solutions (gramme-equivalent dissolved in a litre of water) could all be obtained from the values for a standard solution such as that of LiCl by the addition of certain numbers or moduluses representing invariable differences for the metals and Li and for the negative radicals and Cl. Other properties of solutions have since been proved to be modular, as for instance their heats of formation from their elements and their electric conductivities. I think the modular nature of some of these properties of solutions is the outcome of this modular property in the parameter reciprocal of molecular force along with the additive property in mass. To prove this in the case of density would require a special investigation, but if we assume the property for density we can easily deduce Valson's result that the property applies to capillary elevation. Let h be the height to which a normal solution of any salt RQ rises in a tube of radius 1 millim., then

$$h=2\alpha/\rho=2\mathrm{X}\rho^{\frac{1}{3}}\left(\frac{w+np}{1+n}\right)^{\frac{1}{3}}\Big/ c.$$

Let r and q be the density moduluses of the radicals R and Q, being small fractions, then $\rho=d+r+q$, where d is a constant nearly 1 ; also

$$\mathrm{X}^{-1}=(\mathrm{W}^{-1}+n\mathrm{A}^{-1})/(1+n),$$

so that

$$h=2\left(\frac{1+n}{\mathrm{W}^{-1}+n\mathrm{A}^{-1}}\right)(d+r+q)^{\frac{1}{3}}\left(\frac{w+np}{1+n}\right)^{\frac{1}{3}}\Big/ c.$$

Remembering that in the case of a normal solution n is small, being 18/1000, we can develop the last expression in powers of n as far as the first; and it is evident that as p the molecular mass possesses the additive property and A^{-1} possesses the modular property, then h must also possess the modular property, which is Valson's result.

16. *Second method of finding the Virial Constant for Inorganic Bodies or Solid Bodies in general from the properties of their solutions.*—In this method the compressibility of solutions is used. If a solution could be treated as an ordinary liquid we might attempt to apply the equation of our second method for liquids, namely,

$$l=\frac{4}{3}\left(v_0\frac{a}{\mu}+\frac{25}{26}R\right)vT;$$

but as the solutions to be dealt with are all aqueous, and as a for water at ordinary temperatures is quite abnormal, it would be useless to attempt to apply this equation to them; but from the "dimensions" of the physical quantities involved in it, we may make it yield a correct form of empirical equation for solutions. If we neglect the small term $25R/26$, and also the difference between v and v_0 at ordinary temperatures, the equation above suggests the simple form l varies as $1/\mu\rho^2$, that is X^{-1} varies as $\mu\rho^2$, say $KX^{-1}=\mu\rho^2$, where K is a constant, and for water $KW^{-1}=\mu_w$ the compressibility of water, and as before we have $X^{-1}=(W^{-1}+nA^{-1})/(1+n)$. But on account of the rapid alteration of the compressibility of water with pressure, and its anomalous variation with temperature, we must be prepared to admit that the part of the compressibility of a solution due to the water in it is altered from its value in pure water, and is more altered the more the water is compressed in the process of dissolving the salt. Let this compression be measured roughly by the total amount of shrinkage that ensues when 1 molecule of salt is dissolved in 1000 grms. of water, call the shrinkage Δ, and let us amend the equation given above to $KX^{-1}=\mu\rho^2+f(\Delta)$. Let suffixes a and b attached to symbols refer them to two different bodies, then

$$KX_a^{-1}-KX_b^{-1}=\mu_a\rho_a^2-\mu_b\rho_b^2+f(\Delta_a)-f(\Delta_b),$$

but

$$KX_a^{-1}-KX_b^{-1}=Kn(A_a^{-1}-A_b^{-1})/(1+n),$$

$$\therefore\ Kn(A_a^{-1}-A_b^{-1})/(1+n)=\mu_a\rho_a^2-\mu_b\rho_b^2+f(\Delta_a)-f(\Delta_b).$$

Hence, selecting pairs of bodies for which $\Delta_a=\Delta_b$ approximately, we ought to get $\mu_a\rho_a^2-\mu_b\rho_b^2$ proportional to $cA_a^{-1}-cA_b^{-1}$, the values of the last expression being obtainable from Table XXXI.

To facilitate the comparison I furnish the following broad statements about Δ founded on the study of data as to the molecular volumes of salts, both solid and in solution, given by Favre and Valson (*Compt. Rend.* lxxvii.) Long (Wied. *Ann.* ix.), and Nicol (Phil. Mag. xvi. and xviii.). The modular property applies approximately to shrinkage on solution; the shrinkage of a gramme-molecule of LiCl is 2, and the shrinkage for a gramme-molecule is increased when for Li is substituted

	K,	Na,	NH_4,	½Ca,	½Sr,	½Ba,
by	8	7	−5	10	11	12;

and when for Cl is substituted

	Br,	I,	NO_3,	½SO_4,	½CO_3,
by	0	0	0	8	14.

After giving the values of $\mu\rho^2$ in the next table we can proceed with the comparison.

TABLE XXXIV.—Values of $10^7\mu\rho^2$.

	Cl.	Br.	I.	NO_3.	½SO_4.	½CO_3.
Li	432	484	541	457	433	
Na	441	496	550	468	444	428
K	454	507	557	480	450	437
NH_4	443	493	544	466	442	
½Ca	464					
½Sr..................	497					
½Ba	521					

The experimental data used are those of Röntgen and Schneider (Wied. *Ann.* xxix.), and those of M. Schumann (Wied. *Ann.* xxxi.) for $CaCl_2$, $SrCl_2$, and $BaCl_2$, in the case of which I calculated the compressibility for the half-gramme molecule according to his result that $\mu-\mu_w=cp$ where p is percentage of salt, using a value of c got from the more concentrated solutions.

An inspection of this table shows that $10^7\mu\rho^2$ possesses the modular property; it gives for instance the following differences for Na and Li, 9, 12, 9, 11, 11, with a mean value 10, and so on for the other metals; the mean values for the metals minus that for Li are :—

Na.	K.	NH_4.	½Ca.	½Sr.	½Ba.
10	20	8	32	65	89 ;

and for the negative radicals minus that for Cl :—

Br.	I.	NO_3.	½SO_4.	½CO_3.
52	105	25	0	−15.

We can now select pairs of bodies for which $\Delta_a = \Delta_b$ and test if $\mu_a\rho_a^2 - \mu_b\rho_b^2$ is proportional to $cA_a^{-1} - cA_b^{-1}$.

The following are pairs of elements characterized by nearly equal shrinkage on solution with the values of the differences of $10^7\mu\rho^2$ and of cA^{-1} and also their ratio.

TABLE XXXV.

Equal shrinkage pair...	K, Na.	½Sr, ½Ca.	½Ba, ½Ca.	Br, Cl.	I, Cl.	NO_3, Cl.
Mean diff. of $10^7\mu\rho^2$...	10	33	57	52	105	25
Mean diff. of cA^{-1}...	·29	1·05	1·85	1·56	3·28	·81
Ratio of differences ...	34	31	31	33	32	31

The agreement here is excellent, as will be seen more clearly if we compare bodies not having equal shrinkage, as K and Li, for which we get the $10^7\mu\rho^2$ difference 20, the cA^{-1} difference ·90 with a ratio 22, or K and NH_4, for which the two differences are 12 and 1·05 with a ratio 11.

The agreement above is a verification of the theory of the compressibility of solutions, here barely outlined, and the equation

$$10^7(\mu_a\rho_a^2 - \mu_b\rho_b^2) = 32\,(cA_a^{-1} - cA_b^{-1})$$

when $\Delta_a = \Delta_b$ nearly constitutes a second method of getting values of cA^{-1}; but we will not use it, as it adds no bodies to our list. It suffices to have partly verified the principles on which the first method is founded by their application to quite another physical phenomenon, and especially the principle involved in the remarkable equation

$$X^{-1}=(W^{-1}+nA^{-1})/(1+n).$$

With the values in Table XXXIII. and that for LiCl, namely, ·83, we can obtain the value of cA^{-1} for any salt whose constituents are to be found in the table, or we can if we like use the actual values in Tables XXX. and XXXI.; we can then calculate M^2/cA^{-1}, which is proportional to M^2l, or $M^2l=CM^2/cA^{-1}$, where C is a constant. To connect the values of M^2l thus found with those previously given absolutely in Table XXV., we must find the value of C/c, which we can proceed to do in the following manner :—

We have seen (Section 14) that we had better regard the molecule of water as doubled relatively to that of ordinary liquids, and as we have shown that the molecules are paired in ordinary liquids the molecules are doubly paired in water; but it was suggested that the second pairing of the pairs was not attended with any alteration in the parameter of molecular force, and that the only effect of the second pairing was to make the radius of the molecular domain of water $2^{\frac{1}{3}}$ as large as if water were an ordinary liquid. And, again, in the case of solutions the surface-tensions have been measured at about 15° C., whereas for comparison with our previous work they ought to have been measured at $2T_c/3$, which for water is about 150° C. At this temperature the value of the surface-tension of water reduces, according to Eötvös, to about ·6 of its value at 15° C. Hence the equation, which treating water as an ordinary liquid and at 15° we wrote $\alpha_w=Ww^{\frac{1}{3}}/c$, ought for double pairing and at 150° to become $\cdot 6\alpha_w=W2^{\frac{1}{3}}w^{\frac{1}{3}}/c$, and similar statements hold for the equation for α; so that values of W, X, and A, as deduced from measurements at 15°, ought to be reduced by the factor $\cdot 6/2^{\frac{1}{3}}$, or 1/2, to give the desired values. Now in the case of homogeneous liquids in the equation $l=c\alpha v^{\frac{4}{3}}/m^{\frac{1}{3}}$ giving l in terms of the megadyne we found a value 5930 for $c/2$, with the megamegadyne as unit of force $c/2=\cdot 00593$; and we can use this same value in the case of solutions after we have halved our values of A, or doubled those of cA^{-1} so far given; hence using the values of cA^{-1} so far given we get $M^2l=\cdot 00593M^2/cA^{-1}$.

Fortunately, a test of this argument is made possible by means of Traube's data for the surface-tension of solutions of certain organic acids and sugars, for which the values of $\cdot 00593M^2/cA^{-1}$ are given in the following Table, as well as values of S found by the relation $M^2l=6S$ (the term $\cdot 66S^2$ being omitted), and also values of S calculated from the dynic equivalents in Table XXVI.

TABLE XXXVI.

	Oxalic Acid. $(COOH)_2$.	Citric Acid. $C_3H_4OH(COOH)_3$.	Glycerine. $C_3H_5(OH)_3$.	Mannite. $C_6H_8(OH)_6$.
M^2l	19·4	43·5	33·5	63·0
$S=M^2l/6$............	3·2	7·2	5·6	10·5
S from dynic equiv.	4·2	9·7	5·0	10·0

	Tartaric Acid. $C_2H_2(OH)_2(COOH)_2$.	Dextrose. $C_6H_{12}O_6$.	Saccharose. $C_{12}H_{22}O_{11}$.
M^2l	38·9	72·1	112·5
$S=M^2l6$	6·5	12·0	18·8
S from dynic equivalents	7·4	9·6	18·1

The agreement between the two sets of numbers is not all that could be desired, but it is good enough to show that the only part of M^2l effective in a solution is the linear term in $M^2l=6S+\cdot 66S^2$; and we have already seen that when the molecules of ordinary liquids pair during liquefaction the term $\cdot 66S^2$ is inoperative in the process, so that there is a certain resemblance in the relations of two paired molecules and those of solvent to those of substance dissolved.

Returning to the inorganic compounds we can now tabulate the absolute values of M^2l, calculated according to the relation $M^2l=\cdot 00593M^2/cA^{-1}$. The manner of calculation is best illustrated by an example, say for KBr; first ·83 is taken as the value for LiCl, and to it are added ·90 and 1·56, taken from Table XXXIII., as the differences for K and Li and for Br and Cl; the square of the molecular weight of KBr is divided by this sum and multiplied by ·00593 to give the tabulated value of M^2l in terms of the megamegadyne. From the value of M^2l thus found the dynic equivalent S is calculated by the relation $M^2l=6S$, which has been seen to be appropriate to values of M^2l obtained from solutions.

TABLE XXXVII.—Values of M^2l and S.

	Cl.		Br.		I.		NO_3.		$\frac{1}{2}SO_4$.		$\frac{1}{2}CO_3$.	
	M^2l.	S.	M^2l.	S.	M^2l.	S.	M^2l.	S.	M^2l.	S.	M^2l.	S.
Li ...	12·5	2·1	18·7	3·1	25·7	4·3	17·1	2·8	10·6	1·8		
Na ...	13·7	2·3	20·8	3·5	28·0	4·7	18·8	3·1	12·8	2·1	8·6	1·4
K ...	18·5	3·1	28·3	4·7	32·4	5·4	23·5	3·9	17·2	2·9	12·8	2·1
NH_4	23·0	3·8	27·0	4·5	31·0	5·2	25·0	4·2	17·0	2·8		
$\frac{1}{2}$Mg	10·4	1·7	17·7	3·0	25·1	4·2	15·6	2·6	10·1	1·7		
$\frac{1}{2}$Ca...	12·0	2·0	19·4	3·2	26·7	4·5	17·3	2·9				
$\frac{1}{2}$Sr ...	14·2	2·4	21·8	3·6	29·3	4·9	19·5	3·2				
$\frac{1}{2}$Ba...	18·8	3·1	26·4	4·4	33·9	5·6	24·0	4·0				
$\frac{1}{2}$Zn...	12·5	2·1	...	...	...	...	...	...	12·6	2·1		
$\frac{1}{2}$Cd...	15·7	2·6	23·2	3·9	30·5	5·1	21·0	3·5	16·2	2·7		
$\frac{1}{2}$Mn	11·5	1·9	...	...	...	...	16·8	2·8	11·7	2·0		

To these may be added the following :—

	$AgNO_3$.	$\frac{1}{2}Pb(NO_3)_2$.	$\frac{1}{2}CuSO_4$.	$\frac{1}{2}FeSO_4$.	$\frac{1}{2}NiSO_4$.
M^2l......	30·6	26·3	11·5	11·4	12·3
S..........	5·1	4·4	1·9	1·9	2·0

	$\frac{1}{2}CoSO_4$.	$\frac{1}{6}Al_2(SO_4)_3$.	$\frac{1}{6}Fe_2(SO_4)_3$.	$\frac{1}{6}Cr_2(SO_4)_3$.
M^2l.........	12·7	8·9	12·6	9·6
S............	2·1	1·5	2·1	1·6

The additive principle holds amongst both sets of numbers except in the case of NH_4; see for instance the following list of the differences of S for the iodides and chlorides :—2·2, 2·4, 2·3, 2·5, 2·5, 2·5, 2·5, 2·5. Now we have already seen that the modular principle applies to cA^{-1} (the modular principle applies when a quantity is given by the addition of moduluses to a constant, the additive principle is a special case of the modular in which the constant is zero) ; the additive principle applies to M the molecular weight : hence it would appear to be a mathematical impossibility that the modular or additive principle should apply also to M^2/cA^{-1}, rigorously ; or, more accurately, if the modular principle applies rigorously to one of the quantities cA^{-1} and M^2/cA^{-1} it cannot apply rigorously to the other : but practically we find such relations amongst the numbers that both are approximately obedient to the modular principle. The case of NH_4 casts some light on the question, for with it cA^{-1} shows the same differences in the values of the chloride, bromide, iodide, and nitrate as with the other positive radicals, while M^2/cA^{-1} does not do so: this case would make it appear that the modular principle applies rigorously to cA^{-1}, but not so rigorously to M^2/cA^{-1}.

But leaving out of the count this case of NH_4, significant as it is, we can find mean values for the differences of the dynic equivalents of all the metals and Li, and of all the negative radicals and Cl; if we can obtain the absolute value of the dynic equivalent of Li and of Cl, we shall have those for all the metals and radicals. Now from the organic compounds we have already got a value 1·3 for the dynic equivalent of Cl, and hence from the value for LiCl we could obtain that for Li. The value tabulated for LiCl is 2·1, but we can obtain a mean value fairer to all the other bodies by subtracting, for example, from the value for KI the mean difference for K and Li, and for I and Cl; in this way we arrive at a mean value 1·9 for LiCl, from which, taking the value 1·3 for Cl, we should get ·6 for Li. But the refraction-equivalents of the halogens are supposed by Gladstone to be a little larger in inorganic than organic compounds; so that in the light of our previous knowledge of a close parallelism between dynic equivalents and refraction-equivalents it might be safer to assume that the dynic equivalents of Li and Cl in LiCl are in the ratio of their refraction-equivalents in that compound, namely, 3·8 and 10·7. According to this assumption the values for Li and Cl come out ·5 and 1·4, which we will adopt as true and use in the calculation of the dynic equivalents of the elements given in the following Table. These are measured of course as before in terms of that for CH_2 as unity, and, again, for comparison there are written along with the dynic equivalents the refraction-equivalents in terms of that for CH_2 as unity, calculated from Gladstone's values (Phil. Trans. 1870).

TABLE XXXVIII.

	Li.	Na.	K.	½Mg.	½Ca.	½Sr.	½Ba.	½Zn.	½Cd.
Dynic equivalent	·5	·8	1·6	·3	·55	·9	1·7	·65	1·22
Refraction-equivalent .	·5	·63	1·06	·46	·68	·90	1·04	·67	·9

	Ag.	½Pb.	½Cu.	½Mn.	½Fe.	½Ni.	½Co.	⅓Al.	⅓Fe.
Dynic equivalent	2·70	2·0	·6	·5	·6	·7	·8	·2	·8
Refraction-equivalent .	2·06	1·63	·76	·8	·81	·69	·71	·37	·93

	⅓Cr.	Cl.	Br.	I.	NO_3.	½SO_4.	½CO_3.	½CrO_4.
Dynic equivalent	·3	1·4	2·7	3·8	2·3	1·35	·5	1·7
Refraction-equivalent .	·7	1·4	2·2	3·6	1·8	1·1	·8	2·4

Again we see a remarkable parallelism between the dynic and refraction-equivalents of the elements and radicals. Of course there are refinements which will yet have to be made in the calculation of the dynic equivalents, but it is not likely that they will make the parallelism seriously closer.

17. *Meaning of the Parallelism between Dynic and Refraction Equivalents, and general speculations as to the volumes of the atoms and their relation to ionic speeds.*—We are now called upon to consider the meaning of this parallelism which has been demonstrated both for organic and inorganic compounds, and we shall be helped thereto by the very simple theory which I have given of the Gladstone refraction-equivalent (Phil. Mag. Feb. 1889), showing that to a first approximation

$$(n-1)u = \Sigma(N-1)U,$$

where n is the index of refraction and u the molecular domain of a substance, N the index of the matter of an atom, and U its volume in the molecule. Hence the refraction-equivalent of an element is the product of the refractivity (Sir W. Thomson's name for index minus unity) of the substance of its atom and the volume of the atom (the volume of the atom being measured in terms of the unit in which the molecular domain, usually called molecular volume, is measured); so that the refraction-equivalent is a function of the two variables only, namely, the volume of the atom and the velocity of light through it. Now we have seen that the expression M^2l, as a whole, in one aspect appears to be not dependent directly on the molecular mass M, seeing that M^2l can be represented in terms of certain quantities which we have called dynic equivalents. Hence, as l is proportional to A in our expression $3Am^2/r^4$ for molecular force, we see that in one aspect molecular force seems to be not directly dependent on the mass of the attracting molecules; and yet, on the other hand, in considering solutions we found that the quantity A asserted its individuality as separate from the whole expression Am^2, so that in another aspect there does appear to be a mass action in the attraction of two molecules. However, regarding M^2l or Am^2 as a whole, the simplest hypothesis we can make about the mutual action of molecules is that it depends most on the size of the molecules. This would make Am^2 a simple function of U; so that the dynic and refraction equivalents would have this in common, that they are both simple functions of U. Suppose, now, that the velocity of light through all matter in the chemically combined state is approximately the same, or that N is approximately the same for all atoms as constituents of compound molecules, then the refraction-equivalents given by Gladstone are directly proportional to the volumes of the atoms in the combined state, and then the parallelism between dynic and refraction equivalents would mean that S is nearly proportional to the volume. It is very interesting, therefore, to inquire briefly whether there is any evidence to prove that Gladstone's refraction-equivalents are proportional to the volumes of the atoms; and I think that in Kohlrausch's velocities of the ions in electrolysis we have such evidence. If different solutions, such as those of KCl, NaCl, or $\frac{1}{2}BaCl_2$, are electrolysed under identical circumstances, then we know, according to Faraday's law, that each atom of K and of Na, and each half-atom of Ba, may be considered to receive the same charge, so that they acquire their ionic speeds under the action of the same accelerating force. Accordingly, the ionic speed characteristic of an atom is reached when the "frictional" resistance to its motion is equal to this accelerating force; hence the "frictional" resistance is the same for all atoms, or rather for all electrochemical equivalents. Now the "frictional" resistance will be a function of the velocity of the atom and of its domain (atomic volume) and of its actual volume as well as of the domain and actual volume of the molecule of the solvent; but if water is the solvent in all cases, the only quantities which vary from body to body are the velocity and the domain and volume of the ions, so that we can say

$$\text{"frictional" resistance} = \phi(\mathrm{V}, u, \mathrm{U}),$$

or

$$\mathrm{V} = \mathrm{F}(u, \mathrm{U}).$$

Now the simplest connexion that one can imagine between the velocity and the domain and volume of the ion is that the velocity will be greatest when the free domain or the difference between the domain and the volume is greatest, or, to be more general, when the difference between the domain and some multiple of the volume is greatest; but if N is the same for all combined atoms, then U is proportional to the refraction-equivalent q. Hence the form of F is such that it contains $u-aq$, where a is a constant. On studying the experimental data I found that a might be considered unity, and that V is a linear function of $u-q$. There is a little difficulty in determining with accuracy the domain of an ionic atom in a solution. Nicol, in his work (Phil. Mag. xvi., xviii.) on the molecular domains of inorganic compounds in solution, has confined his attention almost entirely to differences of domains, making the assumption suitable to his purpose that the molecular domain of water is unaltered in solutions, whereas we should expect that the greater part of the shrinkage occurring on solution of an inorganic crystal happens in the water, which is far more compressible than the crystal.

Accordingly I take the molecular domains of salts in the solid state, as given by Long in his paper on diffusion of solutions (Wied. *Ann.* ix.), as nearer to the true domain when they are in solution than Nicol's values; but to allow to a certain extent for the change of state on solution, I have assumed that in each case the water experiences four fifths of the total shrinkage and the dissolved salt one fifth. This is an arbitrary adjustment, and is of no material importance to the comparison to be made except as showing that the point has not been overlooked. In the following Table are given under u the molecular domains, under q the molecular refractions (Gladstone's), in the next column their differences; under k the specific molecular conductivities determined in highly dilute solution by Kohlrausch and shown by him to be equal to the sum of the velocities of the ions in each case. These are taken from his paper (Wied. *Ann.* xxvi.), with a few additions from an earlier one (Wied. *Ann.* vi.). Under k (calculated) are given values of the conductivity calculated from the equation

$$k = 68 + 2{\cdot}2\,(u - q),$$

expressing the linear relation between conductivity or sum of ionic velocities and $u-q$.

TABLE XXXIX.

	u.	q.	$u-q$.	k.	k (calc.).
KI ……………	53·5	35·3	18	107	108
KBr …………	44	25	19	107	110
KCl……………	36	18·8	17	105	105
NaI……………	41	32	9	87	88
NaBr …………	32	21·7	10	87	90
NaCl …………	24	15·5	8·5	87	87
LiCl …………	20·5	14·5	6	78	81
$\frac{1}{2}MgCl_2$ ………	20	14	6	80	81
$\frac{1}{2}CaCl_2$ ………	22	16	6	81	81
$\frac{1}{2}SrCl_2$ ………	24	17·5	6·5	83	82
$\frac{1}{2}BaCl_2$ ………	24	18·6	5·4	86	80
$\frac{1}{2}ZnCl_2$ ………	23	15·8	7	77	83

The agreement is here such as to prove a true connexion between conductivity and $u-q$, the more striking as no relation can be seen between conductivity and u or q taken separately. The only bodies I have omitted from Kohlrausch's latter list are the nitrates of some of the above metals and of silver, the hydrogen compounds of the halogens, and the ammonium compounds. These do not give results in harmony with those in the last table, and, indeed, we should hardly expect a compound radical like NO_3 to experience frictional resistance in the same manner as a single atom like Cl, and as to the hydrogen compounds they form a class by themselves with respect to many physical properties. It will be as well to show the amount of departure in these cases in the following Table :—

	u.	q.	$u-q$.	k.	k (calc.).
HI	56	26	30	327	134
HBr	50	16	34	327	142
HCl	42	11	31	324	136
KNO_3	47	22	25	98	123
$NaNO_3$	36	19	17	82	105
NH_4NO_3	47·5	25·5	22	98	116
NH_4Cl	35·5	22·2	13	104	97

Kohlrausch has pointed out that there is some difficulty in determining the true connexion between ionic velocities and conductivities in the case of the bibasic acids SO_4 and CO_3, so that we must leave them out of the count for the present.

18. *An Attempt to Determine the Velocity of Light through the substance of the water-molecule.*—In spite of the exceptions, the relation demonstrated in Table XXXIX. is sufficiently striking. To explain it, let us replace q by its value $(N-1)U$; then, in interpreting the expression $u-(N-1)U$ as occurring in our expression for the conductivity of a solution, there are two methods of procedure : first, we can assume that $u-U$, the free or unoccupied part of the domain, is the most likely to occur, in which case $N=2$; second, that $u-cU$ occurs in the expression for conductivity, and that c happens to have the same value as $N-1$, on which supposition it would be desirable to determine N. At present I know of only one way of attempting to find N or v/V, the ratio of the velocity of light through free æther to its velocity through the matter of an atom, namely by means of Fizeau's experiment, repeated by Michelson and Morley, on the fraction of its motion communicated by flowing water to light passing through it. Exactly in the manner of my paper (Phil. Mag. Feb. 1889, p. 148) we can estimate the effect of the motion of matter on the light passing through it. Let s be the distance travelled by light in water flowing through the æther at rest with

a velocity δ in the same direction as the light, v'' the mean velocity of light through the flowing water, v' the mean velocity through still water, v its velocity through free æther, V through a molecule of water, l the mean distance through a molecule, and a its mean sectional area ; then the total loss of time experienced by a wave of unit area of front or a tube of parallel rays, or, briefly, a ray of unit section in passing through the matter-strewn path s instead of a clear path in free æther, will be equal to its loss in a molecule multiplied by the number of molecules passed through in the path. This number, when the matter is at rest, is proportional to s, to a, and to ρ/M, or it varies as $sa\rho/\mathrm{M}$; but when the matter is in motion it is reduced in the ratio $1-\delta/v'' : 1$. The loss of time in each molecule is found thus : l/V is the time taken to pass through a molecule, and in this time the molecule moves a distance $l\delta/\mathrm{V}$ and the unit wave-front moves a distance $l(1+\delta/\mathrm{V})$, which in free æther would take a time $l(1+\delta/\mathrm{V})/v$; so that the loss of time in a molecule is $l/\mathrm{V}-l(1+\delta/\mathrm{V})/v$. Hence, the total loss of time in the path s may be written

$$\frac{ksal\rho}{\mathrm{M}}\left(\frac{1}{\mathrm{V}}-\frac{1+\frac{\delta}{\mathrm{V}}}{v}\right)\left(1-\frac{\delta}{v''}\right).$$

But the loss of time is also $s/v''-s/v$. Equating the two expressions and putting $\mathrm{M}/\rho=u$ and $al=\mathrm{U}$, $v/v''=n''$, $v/\mathrm{V}=\mathrm{N}$, we get

$$u(n''-1)=\mathrm{U}\left(\mathrm{N}-1-\frac{\delta}{v}\mathrm{N}\right)\left(1-\frac{\delta}{v}n''\right):$$

as k is equal to 1, seeing that if $\delta=0$ and $\mathrm{U}=u$, then N must be equal to n. This equation is the companion to that for still matter, namely,

$$u(n-1)=\mathrm{U}(\mathrm{N}-1).$$

But to allow for deformation of the wave-front in passing through molecules it was shown (Phil. Mag. Feb. 1889, p. 150) that this first approximation might be altered to the form

$$u(n-1)=\mathrm{U}(\mathrm{N}-1)+c\rho,$$

where c is a constant, and this form was verified, so that we may write

$$u(n''-1)=\mathrm{U}\left(\mathrm{N}-1-\frac{\delta}{v}\mathrm{N}\right)\left(1-\frac{\delta}{v}n''\right)+c\rho$$

$$=\mathrm{U}(\mathrm{N}-1)\left(1-\frac{\delta}{v}\cdot\frac{\mathrm{N}}{\mathrm{N}-1}-\frac{\delta}{v}n''\right)+c\rho,$$

neglecting the term in δ^2;

$$\therefore\ u(n''-n)=-\frac{\delta}{v}\mathrm{U}(\mathrm{N}-1)\left(\frac{\mathrm{N}}{\mathrm{N}-1}+n''\right)$$

$$\therefore\ \frac{n''-n}{n}=-\frac{\delta}{v}\,\frac{\mathrm{U}(\mathrm{N}-1)}{u(n-1)}\cdot\frac{n-1}{n}\left(\frac{\mathrm{N}}{\mathrm{N}-1}+n''\right).$$

Now

$$\frac{n''-n}{n}=\frac{v'-v''}{v''}=-\frac{x\delta}{v''}=-\frac{x\delta}{v'}\text{ approximately,}$$

where x is the fraction of the water's velocity imparted to the velocity v' to change it to v''. Fizeau (*Ann. de Ch. et de Phys.* sér. 3, t. lvii.) found a value ·5 for x, while Michelson and Morley (Amer. Journ. Sc. ser. 3, vol. cxxxi.), in a more extended and accurate series of experiments, found a value $x=\cdot43\pm\cdot02$, which we will adopt. $\mathrm{U}(\mathrm{N}-1)$ is equal to $u(n-1)$ measured in the vapour of water, for which Lorenz (Wied. *Ann.* xi.) gives the value 5·6; the value for water at 20° C., according to his data is 6, and n is 1·333, which may also be taken as the value for n'' where it occurs; all these values being substituted in the equation

$$\frac{\mathrm{N}}{\mathrm{N}-1}=x\frac{n^2}{n-1}\cdot\frac{u(n-1)}{\mathrm{U}(\mathrm{N}-1)}-n''$$

give the value $\mathrm{N}=9$. Hence the velocity of light through the water molecule appears to be one ninth of that through free æther. But before we could ascribe any degree of accuracy to this estimate we should need to be surer of the value of x, whose measurement is attended with great experimental difficulties. It is much to be desired that we had similar measurements for other bodies than water, both liquid and solid, to permit of other estimates of N, so as to see if it is the same for all compound bodies, and also to decide between the theory here sketched and Fresnel's hypothesis, that matter carries its own excess of æther with it, so that $x=(n^2-1)/n^2$, which in the case of water is ·437, in excellent agreement with Michelson and Morley's experimental number; but one such agreement is not sufficient to establish an hypothesis founded on such artificial grounds. However, if $N=9$ then $N-1=8$, and we have the electrical specific molecular conductivity $k=68+2\cdot2\ (u-8U)$. It is only a coincidence that this agrees so exactly in form with Clausius's calculation that the number of encounters experienced per second by a molecule of volume U moving amongst a number of others of volume U is greater than that experienced by an ideal particle moving under the same circumstances in the ratio $u:u-8U$. Further experiment must elucidate the subject-matter of these speculations.

19. *Suggested Relation between the Change in the Volume of an Atom on Combination and the Change in its Chemical Energy.*—Returning to the idea that the dynic equivalent furnishes a measure of the volume of its atom, we can get a suggestive glimpse into the relation between the volume of an atom and its chemical energy. Kundt has recently (Phil. Mag. July 1888) shown that the velocities of light through the metals (uncombined) are as their electrical conductivities, being in the case of silver, gold, and copper greater than through free æther, and as in this case both $n-1$ and $N-1$ are negative, we see that $(n-1)u$ or $(N-1)U$ for the metals changes greatly when the metals pass from the combined to the free state. Now this is in strong contrast to the behaviour of the non-metallic elements, which have been shown in the case of O, N, C, S, P, Cl, Br, and I to possess nearly the same values of $(n-1)u$ in the combined and free states, and the same may perhaps be said of H. Again, in contrast to this approximate inalterability of $(n-1)u$ for these non-metals we have the fact, already pointed out, that the dynic equivalents of H, O, and N are much smaller in the free than the combined state. If, then, the dynic equivalents give a measure of the volumes of the atoms in both states, we must consider the volumes of free H, O, and N to be smaller than when they are combined, the change of volume corresponding to the change of energy on combination. If this is true, then the elasticity and density

of the non-metallic atoms (or the equivalents of these properties in the electromagnetic or any other theory of light) are so related that although the density changes $(N-1)U$ remains constant, whereas in the metallic atoms the relation between density and elasticity must be quite different, because, as we have seen, $(N-1)U$ actually changes sign in some cases.

It would be possible to determine approximate values for the dynic equivalents of the uncombined metals from Quincke's data for the surface-tension of melted metals, and also to get some light on the constitution of salts from his measurements of the surface-tension of melted salts, but these would be most appropriately discussed in connexion with a general study of the elastic properties of solids. I have, however, satisfied myself that the dynic equivalents of the uncombined metals are different from their values in the combined state.

To show the existence of an intimate relation between dynic equivalents and chemical energy we can enumerate the following propositions:—That in the great majority of inorganic compounds the evolution of heat accompanying the passage of an atom from the uncombined to the combined state is almost independent of the nature of the atoms it combines with, similarly the change of dynic equivalent of an atom on combination is almost independent of the nature of the atom it combines with; that in organic compounds, with the exception of the simpler typical forms, the same proposition as this applies both as regards heat and dynic equivalent.

These general remarks are intended to indicate the most hopeful direction for the continuation of these researches to open up new fields; and yet in old fields there is abundance of scope for the application of the law of molecular force towards the acquisition of a knowledge of the structure of molecules, in the elasticity of solids, in the viscosity of gases and of liquids, in the kinetics of solutions, and many kindred subjects.

Melbourne, February 1890.

DISCUSSION (October 28, 1892).

The Discussion on Mr. Sutherland's paper, "*The Laws of Molecular Force*," was reopened by Prof. Perry reading a communication from the President, Prof. Fitzgerald.

He objected to discontinuous theories, especially when Clausius had given a continuous formula much more accurate over a very long range than Mr. Sutherland's discontinuous one. The introduction of Brownian motions without carefully estimating the rates required and energy represented, and without giving any dynamical explanation of their existence, was not satisfactory. It would, he said, be most interesting if Mr. Sutherland would calculate the law of variation of temperature with height of a column of convectionless gas, under conduction alone (for Maxwell thought the inverse fifth power law of molecular attraction was the only one that gave uniformity of temperature under these conditions), and, if necessary, make tests with solid bars. Referring to the statement that molecular attraction at 1 cm. was comparable with gravitation at the same distance, he thought Mr. Boys would question this, and he suggested an *experimentum crucis* of the inverse fourth power law. Both the inverse fourth and inverse fifth power laws assumed symmetry which did not exist. He also took exception to other parts of the paper.

Mr. S. H. Burbury said that, on referring to the author's original paper on which the present one was based, he found that uniform distribution of molecules was assumed. On this supposition the demonstrations given were quite correct, and the potential was a maximum. If, however, the molecules were in motion, the average potential must be less than the maximum, and the deductions in the present paper being based on wrong assumptions were liable to error.

Prof. Ramsay remarked that many statements in the paper on the subject of critical points were very doubtful. Separate equations for the different states of matter were not satisfactory, neither was the artificial division of substances into five classes. The predicted differences in the critical points due to capillarity had not been found to exist. Speaking of the virial equation, he said that hitherto R had been taken as constant. Considerations he had recently made led him to believe that R was not constant. The whole question should be reconsidered, regarding R as a variable.

Mr. Macfarlane Gray said he had been working at subjects similar to those dealt with in Mr. Sutherland's paper, but from an opposite point of view, no attraction being supposed to exist between molecules. In the theoretical treatment of steam he found that no arbitrary constants were required, for all could be determined thermodynamically. The calculated results were in perfect accord with M. Cailletet's exhaustive experiments except at very high pressures, and even here the theoretical volume was the mean between those obtained experimentally by Cailletet and Battelli respectively.

Prof. Herschel pointed out that Villarçeau had discussed the equation of the virial, where the chemical and mechanical energies were not supposed to balance each other. Mr. Sutherland's paper all turns on the existence of such a balance, and he (Prof. Herschel) could not understand why this balancing was necessary.

Remarks by Dr. GLADSTONE.

I was naturally attracted to this elaborate and interesting paper by Mr. Sutherland's frequent references to my refraction work, and to the close relation which he sets forth between Dynic and Refraction Equivalents. Dr. Perkin and I had pointed out a similar relation between molecular refraction, dispersion, and magnetic rotation; and I was much interested in the prospect of a fourth series of analogous values. The four series are arranged in the following table on the same principle as in Mr. Sutherland's Table XXVIII., the value of CH_2 being taken as unity.

Table of Relative Equivalents.

	Dynic.	Refraction.	Dispersion.	Magnetic rotation.
CH_2	1·00	1·00	1·00	1·00
H	0·215	0·17	0·12	0·25
C	0·57	0·66	0·77	0·50
CO″O′	1·9	1·47	1·60	0·97
O′	0·6	0·37	0·29	0·26
NH_2	1·23	1·02	1·35	1·20
CN	1·35	1·33	1·13	1·01
NO_3	2 2	2·0	2·8	0·84
S′	1·6	1·9	3·5	
Cl	1·3	1·3	1·5	1·7
Br	1·6	2·0	3·6	3·5
I	2·3	3·2	7·7	7·6

A glance at this table will show that there is a certain proportionality between the four columns, but with differences which are certainly beyond the limits of experimental error; and the value of the figures is seriously impaired by the fact that while hydrogen is sometimes reckoned at 0·215 by what appear to be somewhat arbitrary assumptions, in others of his calculations Mr. Sutherland looks upon it as so small as to be entirely negligible. But in the case of the three optical properties, the analogy does not depend upon the proportionality of the numbers so much as upon the fact that the refraction, dispersion, or magnetic rotation of a compound is the sum of the refraction, dispersion, or magnetic rotation of its constituent atoms, modified to some extent by the way in which they are combined; and the correspondence is so close in this respect that, in the words of Dr. Perkin and myself, "we find that when a change occurs in the one property it is noticeable also in the other two, and these changes are in the same direction, though not to the same extent." To take one instance out of many, the first and perhaps the second term of a homologous CH_2 series is found to be somewhat out of range with the higher terms in respect of each of the optical properties. Now it would appear that this is also the case in respect of the Dynic equivalent. But the exaltation of effect produced by "double linking" of carbon atoms, which is so very marked in the case of refraction, and still more so in that of dispersion and magnetic rotation, is scarcely apparent in Mr. Sutherland's Dynic equivalents, and he has entirely disregarded it in his calculations.

If we were to take M^2l, instead of the Dynic equivalent which is deduced from it, we should get different values for each increment of CH_2 in a homologous series, increasing as the series ascends; and the same would apply to other radicals. As we deal with the molecular weight M, and not the square of it, in determining the optical properties, it seemed worth while to reckon what constants we should obtain by the formula Ml. When so reckoned, the value for CH_2 would appear to be the same throughout a homologous series, and also in different series. But it necessitates starting each series of compounds with an initial constant H_2, $O''O$, HCl, Cl_2 &c.; and these constants are of high value. Dr. Perkin was led to adopt the same idea of initial constants in investigating the molecular magnetic rotation. By this method of calculation the effect of "double linking" and of "ring formation" becomes very apparent; but it upsets all proportionality between the Dynic equivalents and the Optical equivalents both in regard to the substances given in the preceding table, and the metals contained in dissolved salts.

Regarding the matter from this point of view, I am disposed to believe that Mr. Sutherland's virial constant (l) does represent, more or less correctly, a real property of bodies, and that there is some relationship between it and the three optical properties.

The speculations as to the volumes of the atoms and their relation to ionic speed are very suggestive, and Mr. Hibbert and I hope to make them the basis of some future investigation.

Discussion on Mr. Sutherland's Paper on the Laws of Molecular Force. *By* SYDNEY YOUNG, *D.Sc.*

Mr. Sutherland's calculations and speculations are based on his law of Force between two similar molecules,

$$3Am^2/r^4,$$

together with Clausius' equation of the virial,

$$\tfrac{3}{2}pv=\Sigma\tfrac{1}{2}mV^2-\tfrac{1}{2}\,.\,\tfrac{1}{2}\Sigma\Sigma Rr.$$

Mr. Sutherland first discusses the law established by Amagat for gases and by Ramsay and Young for gases and liquids, that at constant volume $\partial p/\partial T$ is independent of temperature, or, generally, that it is a function of volume only. While admitting the truth of this law for elements and typical compounds such as methane, Mr. Sutherland thinks that for compounds in the liquid state $\partial p/\partial T$ is slightly variable with the temperature; but Barus has shown that with several liquids, including ether, such variability only becomes evident at extremely high pressures.

Mr. Sutherland then points out that the equation of the virial may be written in the form

$$pv = \mathrm{RT}vf(v) + v\phi(v),$$

where $v\phi(v)$ stands for the internal virial term, which, according to the law of the inverse fourth power, should vary inversely as the volume, or

$$v^2\phi(v) \text{ ought to be constant.}$$

But in the case of ether $v^2\phi(v)$ is not constant, but diminishes from a supposed limiting value at large volumes to about half this value near the critical volume, below which there is little change. Mr. Sutherland, however, adheres to his law of the inverse fourth power and attempts to explain the discrepancy by the formation of pairs of molecules at small volumes.

A similar variability is to be noticed in the case of carbon dioxide, but with methane and nitrogen, and presumably the other elementary gases, $v^2\phi(v)$ is considered to be constant, and no pairing is therefore supposed to occur. It may be noticed, however, that the range of volume with carbon dioxide and nitrogen is small, and that in the latter case the values of $v^2\phi(v)$ are very irregular, while methane is a substance that is very difficult to obtain in a state of perfect purity.

Mr. Sutherland finds it necessary to divide substances into at least five classes :—

1. Elements and typical compounds such as methane (no pairing of molecules).
2. Ethylene (intermediate between 1 and 3).
3. Ordinary compounds (molecules pair at small volumes).
4. Alcohols and water (the pairs pair again).
5. Dissociable substances such as nitrogen peroxide and acetic acid (chemical polymerization ; not considered).

I will refer later to the nature of this "pairing" of the molecules.

Mr. Sutherland then discusses the two virial terms and, beginning with ether and carbon dioxide, shows that from large volumes to near the critical volume each term can be represented by a comparatively simple formula, and from these he constructs the characteristic equations. For smaller volumes a different equation has to be employed, and a third or intermediate equation is also required.

It is thus necessary to employ three or at least two characteristic equations for each group of substances; and it may be noticed in passing that for ethylene, which is considered to be intermediate between elements and compounds, an additional constant has to be introduced. It is also somewhat curious that while, with this substance, the pairing is only partial at the critical volume, it does not appear to proceed any further at smaller volumes.

Formulæ are then given by which the critical temperatures, pressures, and volumes may be calculated from the constants employed in the characteristic equations, and a comparison between the observed and calculated values is given. For ethyl oxide these are as follows:—

	Observed.	Calculated.	Δ
Temperature .	194°	199°	5°
Pressure . . .	27·1	29·1	2·2 metres.
Volume . . .	(3·7)	4·7	1 c. c.

Mr. Sutherland seems to consider these differences of temperature and pressure (he does not discuss the errors in volume) as allowable, and here and elsewhere he speaks of an error of 10° or even 20° in the critical temperature as something possible. He also states that capillarity must influence the results. Now this question of the accuracy attainable in the determination of the critical constants appears to me to be one of extreme importance, and I may perhaps refer to some recent work by Mr. G. L. Thomas and myself on the esters. Up to the present we have determined the critical constants of five of these bodies and we have in each case employed two different specimens, great care being of course taken in their purification. The greatest difference we have observed in the critical temperatures of the two samples of the same ester is 0·1° and the greatest difference in pressure is 80 millims.

In the case of ethyl acetate, tubes of different bore were employed in the two determinations, but no effect of capillarity on the critical constants was observable. The actual results for ethyl acetate are:—

	Sample A.	Sample B.
Boiling point (mean)	77°·12	77°·13
Specific gravity at 0°	·92434	·92438
Critical Temperature	250°·1	250°·1
Critical Pressure	28866 mm.	28891 mm.
Internal diameter of tube .	2·04 mm.	1·54 mm.

I think, therefore, that the differences between the observed and calculated results in the case of ether are altogether outside the limits of experimental error, and that either the formula or the constants are not sufficiently satisfactory. With carbon dioxide the differences are much greater, 20° and nearly 20 metres pressure, and I am sure that M. Amagat would not admit any such errors. The critical volume is very much more difficult to determine directly; but in a paper read before this Society about a year ago I showed how the relative critical volumes of different bodies could be ascertained from the data at lower temperatures, and I then thought that in a few cases the observed critical volumes might be taken as fairly accurate.

Since then M. Mathias has shown that the method of MM. Cailletet and Mathias may be employed with great advantage for the determination of the critical densities of the substances referred to in my paper, and I have much pleasure in acknowledging the accuracy both of the method and of the densities (except, perhaps, in the case of the alcohols) calculated by M. Mathias. Further confirmation of the truth of the law on which the method of MM. Cailletet and Mathias is based has been afforded by M. Amagat with carbon dioxide, and by Mr. Thomas and myself with the five esters alluded to.

In part (6) of his paper Mr. Sutherland discusses Van der Waals' generalizations regarding corresponding temperatures, pressures, and volumes, and the results he arrives at are in many respects identical with those which I described to the Society last year. As regards the vapour-pressures at corresponding temperatures, we both conclude that for ordinary compounds the law can only be looked upon as a first approximation, but that when the alcohols are compared with other bodies it cannot be regarded as true at all.

Mr. Sutherland does not, however, discuss the question of volumes at any length.

The last point I wish to allude to is the nature of the supposed "pairing" of the molecules in ordinary compounds, but before doing so I must refer to the internal virial constant l. Mr. Sutherland shows that this constant may be calculated by five more or less independent methods, and that in the few cases where comparison is possible these different methods give fairly concordant results. He then attempts to show that l, or rather M^2l (M being the molecular weight), depends on the composition of the compound and may be calculated by means of what he calls the dynic equivalents, S, of which the relation to M^2l is given by the equation

$$M^2l = 6S + {\cdot}66S^2.$$

Mr. Sutherland gives values of S for several elements and groups. By adding together the values of S for the atoms or groups of atoms in the formula, the dynic equivalent of the compound is obtained, and from this the value of M^2l and of l may be calculated, for

$$l = (6S + {\cdot}66S^2)/M^2.$$

Mr. Sutherland then points out that if in course of compression a true chemical polymer is formed by the pairing of molecules, the new value of l should be

$$l' = (6S/2 + {\cdot}66S^2)/M^2,$$

or l is greater than $\frac{1}{2}\,l$.

But in the characteristic equation for compounds the virial constant changes from the limiting value $\frac{l}{v}$ to $\frac{l}{2v}$ or $l' = \frac{1}{2}\,l$; hence Mr. Sutherland concludes that the pairing in the case of ordinary compounds is not chemical pairing, but he calls it—for the sake of distinction—"physical pairing," and he says "the term $\cdot 66S^2$ would then appear to have a certain chemical significance."

It is, however, surely remarkable that the "physical pairing" should produce a greater effect on the characteristic equation than true chemical pairing, although it has been shown by Dr. Ramsay and myself (Phil. Mag. Aug. 1887, p. 201) that when chemical pairing does take place, as with nitrogen peroxide and acetic acid, the value of $\partial p/\partial T$ at constant volume is no longer independent of the temperature.

If Mr. Sutherland's views are correct, we appear to have several different kinds of pairing of molecules :—

1. Chemical pairing, as with nitrogen peroxide, causing the isochors, or lines of constant volume, to be curved.

2. Physical pairing, with no influence on the isochors, but *more* influence on the internal virial term than chemical pairing.

3. Weaker physical pairing of the physical pairs, as in the case of the alcohols. This has not been further studied, but Mr. Sutherland in part (3) of his paper expresses a doubt whether $\partial p/\partial T$ is really independent of temperature in the case of the alcohols. If, however, this were so, it would appear to follow that curvature of the isochors is caused by true chemical pairing and by weak physical pairing, but not by strong physical pairing.

Mr. Sutherland does not, so far as I can make out, offer any suggestion as to the real nature of this physical pairing, and the idea appears to me to be somewhat speculative and to require further elucidation.

V. *On the Determination of the Critical Density.* *By* SYDNEY YOUNG, *D.Sc., and* G. L. THOMAS, *B.Sc., University College, Bristol*[*].

IT has been shown by M. E. Mathias (*Compt. Rend.* cxv. p. 35) that the method of determining the Critical Density proposed by MM. Cailletet and Mathias (*Compt. Rend.* cii. p. 1202, civ. p. 1563) gives good results in a considerable number of cases. We have recently investigated some of the thermal properties of five of the lower esters, and find that these substances afford further proof of the truth of the "Law of the Diameter" described by these authors. We therefore think it may be well to give the data illustrating this law, without waiting for the completion of the work on the remaining esters.

The law of the diameter may be briefly explained as follows:—It is well known that if the densities of any substance, both as liquid and as saturated vapour, be mapped as ordinates against the temperatures as abscissæ, they fall on a continuous curve which passes through the critical point. This curve possesses, according to MM. Cailletet and Mathias, the characteristic property that the locus of the middle points of chords parallel to the axis of ordinates is a straight line. If this is so, it follows that the ordinate of the point on the diameter corresponding to the critical emperature gives the critical density.

That the law holds very closely for the esters so far investigated is shown by the results tabulated below. The densities of each substance, both in the state of liquid and of saturated vapour, together with the means of these values, are given for a series of temperatures. The mean densities, calculated from the formula $D=A+\alpha t$ (D=density; t= temperature Centigrade; A and α constants depending on the substance) are given in the fifth column, and the differences between these and the observed results in the last column.

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[*] Read October 28, 1892.

Methyl Formate.

Temp.	Densities.				
	Observed.			Calc. Mean.	Δ × 1000.
	Liquid.	Sat. Vap.	Mean.		
40° C.	·9447	·0031	·4739	·4739	0
60	·9133	·0060	·4596	·4596	0
80	·8803	·0105	·4454	·4453	− 1
100	·8452	·0171	·4311	·4310	− 1
120	·8070	·0268	·4169	·4167	− 2
140	·7638	·0412	·4925	·4023	− 2
160	·7136	·0623	·3880	·3880	0
180	·6521	·0943	·3732	·3737	+ 5
190	·6148	·1178	·3663	·3666	+ 3
200	·5658	·1524	·3591	·3594	+ 3
206	·5242	·1862	·3552	·3552	0
210	·4857	·2188	·3522	·3523	+ 1
212	·4549	·2451	·3500	·3508	+ 8
213	·4328	·2681	·3504	·3501	− 3
213·5	·4157	·2865	·3511	·3497	−14
(Critical) 214·0	......	......	......	·3494	

Methyl Acetate.

60° C.	·8801	·0031	·4416	·4425	+ 9
80	·8520	·0056	·4288	·4291	+ 3
100	·8222	·0097	·4160	·4156	− 4
120	·7893	·0158	·4026	·4021	− 5
140	·7533	·0246	·3890	·3887	− 3
160	·7133	·0373	·3753	·3751	− 2
180	·6671	·0568	·3620	·3617	− 3
200	·6100	·0866	·3483	·3482	− 1
210	·5741	·1090	·3416	·3414	− 2
220	·5281	·1415	·3348	·3347	− 1
227	·4818	·1775	·3296	·3300	+ 4
230	·4527	·2030	·3278	·3280	+ 2
232	·4226	·2295	·3260	·3266	+ 6
233	·3995	·2520	·3257	·3259	+ 2
(Critical) 233·7	......	......	......	·3255	

Ethyl Formate.

Temp.	Densities.				
	Observed.			Calc. Mean.	Δ×1000.
	Liquid.	Sat. Vap.	Mean.		
60° C.	·8689	·0034	·4362	·4369	+ 7
80	·8410	·0061	·4236	·4239	+ 3
100	·8112	·0103	·4108	·4102	− 6
120	·7796	·0166	·3981	·3980	− 1
140	·7448	·0256	·3852	·3850	− 2
160	·7058	·0388	·3723	·3720	− 3
180	·6610	·0575	·3592	·3591	− 1
200	·6067	·0862	·3464	·3461	− 3
210	·5724	·1075	·3400	·3396	− 4
220	·5290	·1380	·3335	·3331	− 4
225	·5014	·1585	·3300	·3299	− 1
230	·4635	·1890	·3262	·3266	+ 4
233	·4281	·2198	·3240	·3247	+ 7
234	·4117	·2353	·3235	·3240	+ 5
(Critical) 235·3	......	......	......	·3232	
Ethyl Acetate.					
80° C.	·8245	·0035	·4140	·4144	+ 4
100	·7972	·0062	·4017	·4019	+ 2
120	·7683	·0103	·3893	·3894	+ 1
140	·7378	·0165	·3771	·3769	− 2
160	·7033	·0258	·3645	·3644	− 1
180	·6653	·0388	·3520	·3519	− 1
200	·6210	·0580	·3395	·3394	− 1
220	·5648	·0891	·3270	·3269	− 1
230	·5281	·1130	·3206	·3206	0
240	·4778	·1500	·3139	·3144	+ 5
245	·4401	·1800	·3106	·3103	− 3
247	·4195	·1995	·3095	·3100	+ 5
249	·3893	·2290	·3091	·3088	− 3
(Critical) 250·1	......	......	......	·3081	
Methyl Propionate.					
80° C.	·8408	·0032	·4220	·4224	+ 4
100	·8137	·0057	·4097	·4100	+ 3
120	·7852	·0096	·3974	·3976	+ 2
140	·7553	·0153	·3853	·3852	− 1
160	·7222	·0235	·3728	·3727	− 1
180	·6841	·0355	·3598	·3603	+ 5
200	·6445	·0524	·3484	·3479	− 5
220	·5938	·0781	·3360	·3355	− 5
230	·5635	·0966	·3300	·3293	− 7
240	·5220	·1240	·3230	·3231	+ 1
245	·4975	·1415	·3195	·3200	+ 5
250	·4655	·1675	·3165	·3169	+ 4
253	·4401	·1890	·3145	·3150	+ 5
255	·4151	·2120	·3136	·3138	+ 2
256	·3977	·2295	·3136	·3131	− 5
(Critical) 257·4	......	......	......	·3123	

The constants for the formula $D = A + \alpha t$ are :—

Substance.	A.	α.
Methyl Formate . . .	·5025	—·0007155
Methyl Acetate . . .	·4839	—·0006740
Ethyl Formate . . .	·4759	—·0006490
Ethyl Acetate	·4644	—·0006250
Methyl Propionate . .	·4721	—·0006210

The critical densities of the five esters, obtained by the method of diameters, are :—

Methyl Formate	·3494
Methyl Acetate	·3255
Ethyl Formate	·3232
Ethyl Acetate	·3081
Methyl Propionate	·3123

From the differences between the observed and calculated mean densities it may be concluded that, though there may be in some cases a slight indication of curvature, it is far too small to affect the determination of the critical densities of these substances.

VI. *On the Determination of the Critical Volume.*

By SYDNEY YOUNG, *D.Sc., University College, Bristol**.

ABOUT a year ago I read a paper before the Physical Society † on the "Generalizations of Van der Waals regarding Corresponding Temperatures, Pressures, and Volumes ;" and at the close of the paper, after calling attention to the extreme difficulty attending the determination of the critical volume, I described a method by which the relative critical volumes of various substances could be ascertained. At that time it appeared to me that the direct determinations of the critical volumes of benzene, fluorbenzene, chlorobenzene, and acetic acid were probably approximately correct; and I was supported in this view by the fact that the ratios of the observed volumes to that of fluorbenzene were practically identical with the ratios obtained by the graphical method then described.

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* Read October 28, 1892.

† Proc. Physical Soc. vol. xi. p. 233.

I therefore assumed that the observed critical molecular volume of fluorbenzene was correct, and calculated the volumes of the other substances from their ratios to that of fluorbenzene, and gave these values as the probable critical molecular volumes of the other eleven substances. It was observed, however, that the critical volumes of the three alcohols and of ether obtained by this method were considerably lower than those previously deduced by Dr. Ramsay and myself, and were also lower than those calculated by M. Guye (*Compt. Rend.* cxii. p. 1257). Finally, I pointed out that if these values are taken as correct it would appear that the ratio of the actual critical density to the theoretical (for a "perfect" gas) is for many substances about 4·4.

Since then, M. E. Mathias has shown (*Compt. Rend.* cxv. p. 35) that the critical densities of the substances referred to in my paper may be determined by the method described by MM. Cailletet and Mathias (*Compt. Rend.* cii. p. 1202, civ. p. 1563). The method has also been found by M. Amagat to give excellent results with carbon dioxide, and by Mr. G. L. Thomas and myself with five of the lower esters; and I have therefore much pleasure in acknowledging the correctness both of the method (except, perhaps, in the case of the alcohols) and of the critical densities calculated by M. Mathias, and I have no hesitation in accepting these values in place of those previously adopted.

I have myself made an independent calculation of the critical densities of the twelve substances by this method, and the results I have obtained are practically identical (except in the case of the alcohols) with those given by M. Mathias.

The law of MM. Cailletet and Mathias may be described very briefly as follows:—The means of the densities of any substance in the state of liquid and of saturated vapour, when mapped against the temperature, fall on a straight line. I quite agree with M. Mathias that in the great majority of cases the verification of the law is perfect; but with the three alcohols, especially methyl alcohol, there appears to me to be very decided curvature, too pronounced to be neglected, and I think it must be admitted that these substances behave exceptionally in this as in many other respects.

M. Mathias admits that at low temperatures the alcohols cease to follow the law ; but he is satisfied with the agreement between the observed and calculated mean densities for a range of 200° from the critical point. The constants employed by M. Mathias for ethyl alcohol can be ascertained from his paper, and I give below the mean densities calculated from these constants, together with those calculated from the observed molecular volumes given in my paper (*loc. cit.*).

The constants for the formula $D = A + \alpha t$ are :—

$A = \cdot 4149$, $\alpha = - \cdot 0005576$, t = temperature (Centigrade).

Means of Densities of Liquid and Saturated Vapour.

Temperature, Centigrade.	Observed.	Calculated.	$\Delta \times 1000$.
22·4	·3937	·4024	+87
39·45	·3865	·3929	+64
65·75	·3749	·3782	+33
89·7	·3636	·3649	+16
105·55	·3560	·3560	0
128·1	·3442	·3435	− 7
144·65	·3350	·3342	− 8
151·65	·3309	·3303	− 6
163·8	·3240	·3236	− 4
187·9	·3104	·3101	− 3
206·35	·2998	·2998	0
221·35	·2899	·2915	+16
229·2	·2850	·2871	+21
234·3	·2811	·2843	+32
238·9	·2781	·2817	+36
243·1	......	·2793	

From the above differences it will, I think, be concluded that there is considerable curvature of the "diameter" in the case of ethyl alcohol, and with methyl alcohol it would be still more marked.

I now give a revised table of the critical densities and volumes of the twelve substances, adding, for the sake of convenience, the critical temperatures and pressures. The critical densities are those calculated by M. Mathias ; but in the case of the alcohols I give also the values which I have

Critical Constants.

Substance.	Temperature, Cent.	Pressure, in millims.	Density.	Vol. of 1 gram.	Mol. Vol.	Ratios of Mol. Vol. to that of Fluorbenzene.	
						New.	Old.
Fluorbenzene	286·55	33910	·3543	2·822	270·4	......	......
Chlorobenzene	360	33910	·3661	2·731	306·5	1·133	1·125
Bromobenzene	397	33910	·4857	2·059	322·4	1·192	1·180
Iodobenzene	448	33910	·5838	1·713	348·4	1·289	1·277
Benzene	288·5	36395	·3037	3·293	256·3	·948	·944
Carbon Tetrachloride	283·15	34180	·5558	1·799	276·1	1·021	1·010
Stannic Chloride	318·7	28080	·7423	1·347	349·3	1·292	1·290
Ether	194·4	27060	·2631	3·801	280·7	1·038	1·047
Acetic Acid	321·6	43400	·3514	2·846	170·3	·630	·630
Methyl Alcohol. M.	240·0	59760	·2784	3·592	114·7	·424	·440
,, ,, Y.	,,	,,	·2705	3·697	118·0	·437	
Ethyl Alcohol. M.	243·1	47850	·2793	3·580	164·3	·608	·620
,, ,, Y.	,,	,,	·2750	3·636	166·9	·617	
Propyl Alcohol. M.	263·7	38120	·2778	3·600	215·5	·797	·800
,, ,, Y.	,,	,,	·2752	3·634	217·6	·805	

obtained by the same method, taking into account the curvature of the "diameter." The critical volumes of a gram and molecular volumes are calculated from the densities. The last two columns contain the ratios of the new critical molecular volumes to that of fluorbenzene, together with the ratios given on p. 184 of my previous paper and obtained by the graphical method there described. It will be seen that these ratios show satisfactory agreement.

As regards the alcohols it will be noticed that, if the curvature of the diameter is taken into account, the critical densities are lower and the volumes higher; also that the ratios of the critical molecular volumes to that of fluorbenzene agree better with those obtained by the graphical method.

It may be pointed out that if the generalizations of Van der Waals were strictly true, the angular coefficients (α) of the diameters of different substances should be directly proportional to their critical densities and inversely proportional to their absolute critical temperatures, or that for any substance

$$\alpha = \text{const.} \times \frac{D}{T}; \text{ or } \frac{\alpha T}{D} = \text{const.}$$

I have calculated the values of this "constant" for the substances mentioned in this paper (with the exception of methyl and propyl alcohol) and for the five esters studied by Mr. G. L. Thomas and myself.

Substance.	$\frac{\alpha T}{D}$.	Substance.	$\frac{\alpha T}{D}$.	Substance.	$\frac{\alpha T}{D}$.
Fluorbenzene	·932	Carbon Tetrachloride	·899	Methyl Formate...	·997
Chlorobenzene	·953	Stannic Chloride . .	·988	Methyl Acetate ...	1·047
Bromobenzene	·957	Ether	·972	Ethyl Formate ...	1 061
Iodobenzene	·957	Acetic Acid...........	·993	Ethyl Acetate......	1·055
Benzene	·921	Ethyl Alcohol*	1·030	Methyl Propionate	1·021

It can hardly be said that there is more than a rough approximation to constancy in the values of $\frac{\alpha T}{D}$.

I have recalculated the ratios of the actual to the theoretical critical densities, and the results are given in the following table:—

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* M. Mathias' value of α.

Fluorbenzene	3·79	Stannic Chloride....	3·75
Chlorobenzene	3·78	Ether	3·82
Bromobenzene	3·81	Methyl Alcohol	4·52
Iodobenzene	3·79	Ethyl Alcohol......	4·02
Benzene	3·71	Propyl Alcohol	4·02
Carbon Tetrachloride.	3·66	Acetic Acid........	5·00

The first eight numbers agree fairly well; and it may be stated that *for many substances the ratio of the actual critical density to the theoretical density (for a perfect gas) is about* 3·8. The alcohols, notably methyl alcohol, and acetic acid, form exceptions to this law, and in the case of the esters the ratios (3·89 to 3·95) are rather high.

VII. *On the Boiling-points of different Liquids at Equal Pressures.* *By* SYDNEY YOUNG, *D.Sc., Professor of Chemistry, University College, Bristol*[*].

EARLY in the present year, M. E. Colot (*Compt. Rend.* cxiv. p. 653) described a relation between the boiling-points of any two liquids under equal pressures, which may be shortly described as follows.

If t and θ are the temperatures of the two substances at any pressure, then

$$t = A\theta + B,$$

where A and B are constants depending on the nature of the liquids.

The constants given by M. Colot refer to Centigrade temperatures, but for the purpose of this paper it is better to employ absolute temperatures. In this case A remains unaltered, but B has a different value.

There are certain cases in which $B=0$ (for absolute temperatures), and the equation then becomes

$$t = A\theta \text{ or } \frac{t}{\theta} = A.$$

This is identical with the equation given by Dr. Ramsay and myself (Phil. Mag. Jan. 1886, p. 33) for certain closely related substances such as ethyl chloride and bromide or chlorobenzene and bromobenzene, and I have shown that it holds good for the four halogen derivatives of benzene.

[*] Read October 28, 1892.

For bodies which are not closely related we proposed an equation which may be written

$$R = R_0 + ct,$$

where R is the ratio of the absolute temperatures of the two substances at any pressure p, R_0 is the ratio at such a pressure that the temperature of the second liquid $= 0°$ C., and t is the Centigrade temperature of the same liquid at the pressure p.

The second terms of the two formulæ (B in that of M. Colot and ct in the one just mentioned) are not equivalent, but they both serve to bring the calculated results pretty well within the limits of experimental error, though not in all cases for an indefinite range of pressure.

I have calculated the constants for both formulæ for eleven pairs of liquids of which the boiling-points, at a number of equal pressures, are given in Table I.

The constants for the eleven pairs of substances are given in Table II.

It will be noticed that the constants B and c are always of opposite sign, and that as regards magnitude they both fall, with one exception, in the same order.

In Table III. I have given the differences between the calculated and observed temperatures for each pair of liquids and for both formulæ. I have in each case taken the greatest available range of pressure to make the test as searching as possible.

It will be seen that when the constants B or c are very small, both formulæ give good results, and either of them may be employed for a very wide range of pressure. For the equation $R = R_0 + ct$ the agreement is good in the case of methyl acetate and benzene, although c has a comparatively high value, but when one of the two liquids is water, an alcohol, or acetic acid, the differences are decidedly greater.

The applicability of the formula $t = A\theta + B$ seems to depend more directly on the magnitude of the constant B; there is, indeed, a rough proportionality between the constants and the mean differences given at the foot of Table III. It is quite clear that in the last five cases the relation between the boiling-points is represented by a curve and not a straight line, and a slight tendency to curvature is noticeable even in some of the other comparisons.

TABLE I.

Boiling-Points, Centigrade.

Pressure in mm.	Benzene. C_6H_6.	Fluor-benzene. C_6H_5F.	Carbon Tetra-chloride. CCl_4.	Stannic Chloride. $SnCl_4$.	Ether. $(C_2H_5)_2O$.	Methyl Acetate. CH_3 $\vert$ $COOCH_3$.	Ethyl Acetate. C_2H_5 $\vert$ $COOCH_3$.	Water. H_2O.	Methyl Alcohol. CH_3OH.	Ethyl Alcohol. C_2H_5OH.	Acetic Acid. CH_3 $\vert$ $COOH$.
10	......	−12·35	−20·5	8·8	......	......	−14·0	11·25	−16·25	− 2·3	17·2
50	11·8	+16·3	+ 7·95	39·6	......	−4·05	+12·8	38·3	+ 8·6	+22·2	47·4
100	26·25	30·9	22·25	55·35	−12·1	+9·05	26·35	51·6	20·9	34·5	62·7
200	42·4	47·25	38·7	73·0	+ 1·6	23·45	41·6	66·45	34·5	48·1	79·7
400	60·8	65·75	57·2	93·0	17·4	39·85	59·0	83·0	49·45	62·95	98·7
760	80·25	85·1	76·75	114·2	34·45	57·25	77·1	100·0	64·8	78·3	118·2
1000	89·5	94·3	85·95	124·25	42·45	65·3	85·9	107·95	72·0	85·4	127·35
2000	115·35	120·25	112·3	152·55	65·05	87·9	110·05	129·55	91·6	104·9	152·8
4000	146·0	150·8	143·3	186·0	91·55	114·5	138·25	154·25	113·8	127·2	182·1
7000	175·05	179·8	172·95	217·75	116·6	139·3	164·75	176·9	134·3	147·8	209·0
10000	196·05	200·6	194·4	240·55	134·6	157·3	183·95	192·8	148·8	162·55	227·8
15000	222·1	226·7	220·8	269·1	157·0	179·55	207·75	212·25	166·4	180·75	250·6
20000	242·2	246·7	241·3	290·9	174·55	196·7	225·75	227·3	179·9	194·7	268·15
25000	258·8	263·0	258·3	309·1	189·1	210·7	240·4	239·9	191·0	206·2	282·45
30000	272·9	277·0	272·7	......	......	222·75	......	250·5	200·5	216·05	294·8
35000	285·25	......	285·2	......	......	233·2	......	259·9	208·9	224·7	305·7
40000	......	......	......	......	......	......	......	268·35	216·15	232·3	315·4

TABLE II.

Substances.	For $t=A\theta+B$.			For $R=R_0+ct$.		
	A.	B.	t=absolute temp. of	R_0.	c.	t=Centigrade temp. of
$CH_3COOC_2H_5$ and CH_3COOCH_3	1·0620	− 0·20	$CH_3COOC_2H_5$	1·0611	+·0000035	CH_3COOCH_3
C_6H_6 and $(C_2H_5)_2O$	1·1556	− 2·05	C_6H_6	1·1483	+·0000160	$(C_2H_5)_2O$
C_6H_6 and C_6H_5F	1·0018	− 5·45	C_6H_6	0·9838	+·0000313	C_6H_5F
C_6H_6 and CCl_4	0·9843	+ 8·90	C_6H_6	1·0138	−·0000500	CCl_4
$SnCl_4$ and CCl_4	1·0758	+ 10·70	$SnCl_4$	1·1130	−·0000710	CCl_4
H_2O and C_2H_5OH	1·0953	− 11·70	H_2O	1·0552	+·0000777	C_2H_5OH
CH_3COOCH_3 and C_6H_6	0·8637	+ 24·55	CH_3COOCH_3	0·9459	−·0001410	C_6H_6
$SnCl_4$ and CH_3COOH	1·1349	− 53·55	$SnCl_4$	0·9585	+·0003000	CH_3COOH
C_6H_6 and HO_2	1·2528	−112·85	C_6H_6	0·8885	+·0006130	H_2O
CCl_4 and CH_3OH	1·3872	−115·20	CCl_4	0·9840	+·0008370	CH_3OH
C_6H_6 and C_2H_5OH	1·3649	−124·40	C_6H_6	0·9441	+·0008030	C_2H_5OH

TABLE III.

Differences between Calculated and Observed Temperatures.

Pressure in mm.	$CH_3COOC_2H_5$ and CH_3COOCH_3.		C_6H_6 and $(C_2H_5)_2O$.		C_6H_6 and C_6H_5F.		C_6H_6 and CCl_4.		$SnCl_4$ and CCl_4.		H_2O and C_2H_5OH.		CH_3COOCH_3 and C_6H_6.		$SnCl_4$ and CH_3COOH.		C_6H_6 and H_2O.		CCl_4 and CH_3OH.		C_6H_6 and C_3H_5OH.	
	C.	R.&Y.	C.	R.&Y.	C.	R.&Y.	C.	R.&Y.	C.	R.&Y.	C.	R.&Y.	C.	R.&Y.	C.	R.&Y.	C.	R.&Y.	C.	R.&Y.	C.	R.&Y.
10	......	......	...	...	...	...	...	...	+0·55	−0·4	0	+0·8	......	......	−6·0	−2·05	...	...	−11·55	−3·35	...	...
50	−0·4	−0·4	...	...	−0·45	−0·05	+0·65	−0·1	+0·35	0	+0·35	+0·7	+1·6	−0·05	−2·55	−0·95	−7·65	−0·9	− 5·5	−1·85	−6·0	−0·85
100	0	−0·05	+0·2	+0·3	−0·25	−0·1	+0·25	−0·25	0	−0·2	+0·5	+0·7	+0·95	−0·1	−0·9	−0·25	−5·45	−0·6	− 2·75	−0·9	−3·95	−0·4
200	+0·05	0	−0·1	−0·05	−0·05	−0·05	+0·3	0	+0·05	+0·1	+0·55	+0·55	+0·5	0	+0·75	+0·5	−3·0	0	− 0·35	−0·25	−1·5	+0·15
400	+0·05	0	−0·25	−0·25	−0·1	−0·05	+0·1	0	−0·05	+0·15	+0·25	+0·15	0	0	+2·3	+1·3	−0·65	+0·65	+ 1·9	+0·45	+0·35	+0·35
760	+0·4	+0·4	0	−0·05	+0·05	+0·05	−0·1	0	−0·25	+0·2	+0·1	−0·15	−0·6	−0·1	+3·25	+1·65	+1·2	+1·0	+ 3·65	+0·95	+1·85	+0·5
1000	+0·15	+0·15	−0·05	−0·05	0	+0·05	−0·3	−0·15	−0·4	+0·05	−0·1	−0·4	−0·65	0	+3·55	+1·75	+1·9	+1·2	+ 4·45	+1·35	+2·3	+0·45
2000	0	0	+0·25	+0·15	+0·15	+0·25	−0·2	+0·1	−0·35	+0·2	−0·35	−0·7	−0·95	+0·1	+4·15	+2·1	+3·1	+1·25	+ 6·45	+1·45	+3·05	+0·25
4000	+0·05	+0·1	+0·25	+0·15	+0·1	+0·05	−0·05	+0·35	−0·1	+0·45	−0·6	−1·0	−1·05	+0·2	+3·95	+2·05	+3·4	+1·05	+ 4·8	+0·85	+2·85	−0·3
7000	−0·1	+0·05	+0·15	+0·05	+0·1	+0·05	−0·2	+0·25	−0·3	+0·1	−0·7	−1·05	−0·75	+0·45	+2·75	+1·45	+2·75	+0·45	+ 3·85	+0·6	+1·9	−0·8
10000	−0·2	−0·1	−0·1	−0·1	−0·05	−0·2	−0·1	+0·25	0	+0·2	−0·45	−0·7	−0·65	+0·4	+1·25	+0·65	+1·65	−0·15	+ 2·5	+0·15	+1·05	−1·0
15000	−0·35	−0·3	−0·25	−0·25	+0·05	+0·05	−0·15	+0·1	−0·15	−0·25	+0·05	−0·1	−0·4	+0·25	−1·4	−0·85	0	−0·8	+ 0·55	−0·2	−0·2	−0·85
20000	−0·15	−0·05	−0·05	−0·05	0	+0·1	−0·05	0	+0·1	−0·3	+0·25	+0·3	−0·15	0	−3·3	−1·7	−1·3	−1·0	− 1·25	−0·45	−1·25	−0·55
25000	+0·1	+0·2	+0·15	+0·2	−0·3	−0·1	+0·05	0	+0·15	−0·5	+0·25	+0·4	+0·15	−0·1	−5·25	−2·65	−2·1	−0·65	− 2·85	−0·55	−2·15	0
30000	......	......	...	...	−0·35	−0·05	+0·15	−0·1	...	...	+0·45	+0·75	+0·3	−0·4	...	...	−2·9	−0·35	− 4·05	−0·3	−2·8	+0·65
35000	......	......	...	...	...	...	+0·1	−0·35	...	...	+0·55	+1·0	+0·5	−0·6	...	...	−3·5	+0·15	− 4·9	+0·3	−3·35	+1·3
40000	......	......	...	...	...	...	...	...	...	...	+0·4	+0·7	......	......	...	...	...	...	......	......	...	...
Mean Diffs.	0·15	0·14	0·15	0·14	0·14	0·09	0·18	0·13	0·20	0·22	0·35	0·60	0·61	0·18	2·95	1·42	2·70	0·68	3·83	0·87	2·30	0·56

It is obvious, however, that by taking a smaller range of pressure—say from the lowest to 2000 millim., or from 4000 millim. to the highest pressure, and by altering the constants, a very much better agreement would be obtained. Extrapolation beyond the limits of pressure chosen would, however, introduce very large errors indeed.

In conclusion it may, I think, be stated that in a great number of cases, including pairs of widely different bodies, the formula of Ramsay and Young gives better results than that of M. Colot; the latter, which is very convenient, may, however, be employed, even for very wide ranges of pressure, when the constant B is small, and for small ranges of pressure when B is large.

VIII. *Separation and Striation of Rarefied Gases under the Influence of the Electric Discharge.* *By* E. C. C. BALY[*].

SOME time ago, on examining with a spectroscope a vacuum-tube which happened to contain a small quantity of hydrogen, during the passage of the electric spark I noticed that the hydrogen lines, while strongly visible in the negative glow, could not be seen in the body of the tube. The hydrogen appeared to be, in fact, withdrawn from the tube and collected about the negative pole. Finding the same result in a tube which I fitted up for the purpose, it appeared to me to point to a separation of the gases in the tube; and I determined to make a series of experiments with a view to investigating the matter and the behaviour generally of different gases under similar conditions.

The tubes I employed were about 9 inches long and ¾ inch internal bore. The electrodes were of aluminium wire, and, except in certain cases to which I shall refer, about 1½ inch in length. Two of these tubes were connected to the pump at the same time, one direct and the other through tubes for the absorption of mercury vapour; so that in all the experiments results were obtained in the presence and in the absence of mercury vapour. Gas-generators, fitted with purifying and drying apparatus, were connected so as to allow of varying quantities of the particular gases under examination being admitted to the tubes as might be required. The pump, I may mention, was a modified form of the Geissler mercury-pump, to which I was able to attach an automatic apparatus for working it—a very great saving of labour. Measurements of the vacua were obtained by means of an ordinary barometer-gauge, and varied from 15 millim. to ¼ millim.

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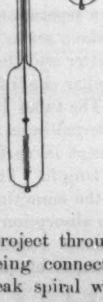
[*] Read February 10, 1893.

I first worked with varying quantities of carbon dioxide and hydrogen. On using certain mixtures of these gases exhausted to about ¾ millim., when the current was first passed a white glow appeared throughout the tube, no striæ being visible, giving a mixed spectrum of the two gases. After a few seconds the negative glow changed to a pink colour, and well-defined striæ, whiter than the preceding glow, began to appear. On watching this change with the spectroscope, the hydrogen lines in the tube were seen to become fainter and gradually to disappear, leaving only the spectrum of carbon dioxide, while those in the negative glow became extremely brilliant. The current was then stopped, and the tube allowed to rest for about an hour, when, the current being passed again, the same phenomenon precisely occurred. Judging that if this were caused by actual separation of the two gases, it ought to be possible to fractionate out the hydrogen into another tube, I endeavoured for a long time unsuccessfully to do so; but at length succeeded by using a double tube of the shape shown in the figure. The longer of the tubes consisted of two chambers joined by capillary tubing, the smaller tube being connected to one of these by a narrow neck capable of being sealed off. Both tubes were furnished with electrodes. One of those in the large tube was sufficiently long to project through the capillary into the second chamber, and being connected at its base to the sealed platinum by a weak spiral wire



could be dropped sufficiently far to touch the opposite electrode. This rod was made of copper, as being heavier than aluminium. It fitted the capillary fairly well, and was furnished with small stops of cotton-wool in order to close the capillary, and thus prevent as much as possible the diffusion of the gases. The tube, after being filled with the gases carbon dioxide and hydrogen, was exhausted and sealed from the pump. The copper rod was caused to touch the opposite electrode, and the current was passed so as to make the whole of the long tube the negative pole and the electrode at the lower end of the smaller tube the positive. The current was continued for a considerable time, when the connecting-neck between the tubes was sealed off and the copper rod shaken back to the stop. On comparing the spectra of the two tubes, it was found that the small tube (containing the positive pole) showed only a trace of hydrogen, while the other showed it brilliantly.

This, I think, may be considered a proof of an actual though not complete separation of the two gases.

I then tried hydrogen mixed with many other gases—amongst them nitrogen, carbon monoxide, sulphur dioxide, iodine, and mercury vapour; and in every case, without exception, I found that the hydrogen was collected about the negative pole in exactly the same manner as I have particularized.

I proceeded to examine mixtures of other gases. With carbon monoxide and carbon dioxide, the carbon monoxide is separated out and appears in the negative glow. With carbon dioxide and nitrogen the carbon dioxide is separated out, while the nitrogen remains in the tube, the separation being remarkably distinct. In this case it is the heavier gas which is separated out; and the same is found with carbon dioxide and sulphur dioxide, the sulphur dioxide appearing in the negative glow.

It would thus appear that the separation of two gases does not depend on their relative molecular weights. On examining a tube of air, however, the components of which are of fairly equal molecular weight, a separation was found to be very difficult, and only occurred after carrying the exhaustion to a much higher point than was usually necessary. The nitrogen remained in the tube, and the negative glow gave a spectrum presumably oxygen; I say presumably, for I am unable to see what else it can be. I was unable to produce the same spectrum in a tube with oxygen, but I was prevented from proceeding further. The spectra of oxygen Professor Schüster has shown to be very varied under different conditions. Is it not possible that the two spectra in an oxygen tube, the banded one in the negative glow, and the bright-line one in the rest of the tube, may be due to separation of two gases?

From these experiments it is evident that when the electric current is passed through a rarefied mixture of two gases, a process similar to electrolysis is set up, one of the gases being separated out and collected about the negative pole, the other gas remaining in the tube; the proof being that the gas separated out may be fractionated into another tube by the method I have above described. In pursuing these experiments I was struck by the apparent close connexion between separation and striation; that is to say, I found strongly marked striæ when there was good separation, and feeble striæ when the separation was difficult; I also found that the first appearances of these phenomena were coincident, the formation of striæ being always the sign of the commencement of separation. There were no exceptions to this, the action in all the tubes I made being the same.

It was evident to me that, if the connexion were real and the separation of the gases could in some way be prevented, by avoiding the negative glow, striæ would not be formed. I accordingly made a tube the negative pole of which did not protrude from the little glass collar in which it was placed, the positive electrode being made as usual. The tube was filled with a mixture of hydrogen and carbon dioxide and exhausted. At a pressure of 24 millim. the current was passed; instead of a negative glow appeared a little bunch of light about $\frac{1}{8}$ inch below the negative point. If this bunch of light impinged on the side of the tube, the glass became strongly phosphorescent. At 4 millim. there was no bunch of light or negative glow; there was no sign of any striæ, the tube giving a spectrum of hydrogen and carbon dioxide, and no evidence of separation. I then reversed the current; immediately striæ formed, separation began and became well-marked.

I tried various mixtures of gases and always obtained the same result, viz., that when the negative glow was avoided by the use of the minute electrode point, neither striæ nor separation occurred, but in reversal of the current strongly marked striæ and good separation.

My next step was to experiment with a pure vapour, which, if my contention be correct, should not striate. It is known that a tube of pure mercury vapour does not stratify. I prepared a tube for the purpose, one end of which was connected with the pump and the other with a bulb containing mercury. After exhaustion, I strongly heated the tube and boiled the mercury, which thus distilled through the tube. On passing the current, as I expected, no striæ appeared, but simply a beautiful phosphorescence throughout the tube, giving a spectrum of pure mercury only. On ceasing to boil the mercury and allowing the tube to cool, a small quantity of other gas was necessarily drawn in from the pump. Immediately striæ began to appear, beginning at the end of the tube connected with the pump; the negative glow changed at the same time and gave a spectrum containing other lines in addition to those of mercury, thus strongly confirming my previous conclusion.

Thinking that possibly the absence of striæ might arise, not from the purity of the mercury vapour, but from its molecules being monatomic, I repeated the experiment with pure vapours of iodine, sulphur, arsenic, and mercuric iodide, which are not monatomic. The result was in each case precisely the same—the tube while heated showing no striæ, but on cooling both striæ and separation.

Wishing if possible to obtain a pure gas which did not stratify at *ordinary* temperatures, I made many attempts to prepare a tube of pure hydrogen. The nearest approach to success was with hydrogen prepared from pure caustic potash and aluminium. The gas was then absorbed by red-hot palladium, which was re-heated in the vacuum. On passing the current, the tube showed an even phosphorescent light throughout, with a very faint line of the most delicate striæ, very difficult to distinguish. The striæ did not become any plainer on carrying the exhaustion to a considerably higher point. I think it may be safely assumed that a tube of perfectly pure hydrogen would not striate at all.

From the foregoing results I think the following conclusions may be drawn :—

First, that when an electric current is passed through a rarefied mixture of two gases, one is separated from the other and appears in the negative glow.

Second, that striæ are caused by the separation of the two gases, and do not occur in a single pure gas or vapour.

DISCUSSION.

Mr. Crookes said :—

In my Inaugural Address delivered before the Institution of Electrical Engineers on January 15th, 1891, I showed some experiments and gave reasons which had led me to believe that the stratifications seen when a high-tension current is passed through a rarefied gas were due to the gas not being homogeneous. I took the case of hydrogen in a vacuum-tube at a pressure of 2 millim. The gas was prepared from zinc and sulphuric acid, and was purified and dried in the usual manner, but with no special precautions against impurities. The exhaustion was effected with a mercury pump, thus leaving mercury vapour in the residual hydrogen.

The stratifications are tri-coloured—blue, pink, and grey. Next the negative pole is a luminous layer ; then comes a dark interval, or Faraday's dark space ; and after this are the stratifications, the front component of each group blue, the next pink, and the third grey. The blue discs are somewhat erratic. At a certain stage of exhaustion all the blue components of the stratification suddenly migrate to the front, forming one bright blue disc, and leaving the pink and grey components by themselves. When the tube contains a compound gaseous residue of this kind, the form of stratifications can be very considerably altered by varying the potential of the discharge. . . . . That the alteration depends simply on the difference of potential, the following experiment pretty clearly shows:—Take a tube giving on the coil the coloured stratifications usually attributed to the presence of residual hydrogen, but which I find is *due to a mixture of hydrogen, mercury, and* *hydrocarbon vapours.* Then, by altering the break so as to produce frequent discharges of lower potential, the stratifications gradually change in shape and become all pink; again, altering the break, so as to send less rapid discharges at a much higher potential, once more we get the coloured stratifications. When in this state, if we introduce a water resistance into the circuit, so as to damp down the potential, exactly the same thing happens. *The blue disk is caused by mercury*; its spectrum is that of mercury only, without even a trace of the bright red line of hydrogen. Experiments not yet finished make it very probable that the pink discs are due to hydrogen, and that the grey discs indicate carbon. The tube contains nothing but hydrogen, mercury, and a minute trace of carbon; but with all the resources at my command I have not been able to get hydrogen quite free from impurity. Indeed I do not think absolutely pure hydrogen has ever yet been obtained in a vacuum-tube. I have so far succeeded as to completely eliminate the mercury, and almost completely to remove the trace of carbon.

IX. *Notes on some Recent Determinations of Molecular Refraction and Dispersion.* *By* J. H. GLADSTONE, *D.Sc., F.R.S.*[*]

AMONG the various indices of refraction which have been published by different observers of late years, there are some which have led me to further studies bearing on the general relation of this branch of Physics to Chemical theory. The following notes relate to the new metallic carbonyls, the metals indium and gallium, sulphur, and to liquefied oxygen, nitrous oxide, and ethylene.

I. METALLIC CARBONYLS.

Messrs. Ludwig Mond and Nasini[†] determined the molecular refraction of nickel tetracarbonyl for the red ray of hydrogen (H_a) as 57·7, and the specific dispersion between the lines γ and *a* of hydrogen as 0·03475, which will give a molecular dispersion between those limits of 5·93. This indicates a very great refraction and an enormous dispersion.

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[*] Read February 10, 1893.

[†] *Lincei, Rendiconti*, vii. 411.

Through the kindness of Mr. Mond I have been able to determine the refraction of two specimens of iron pentacarbonyl for several lines of the spectrum, with the following result. The second and third lines in the Table relate to the same specimen, the latter representing measurements taken in a very acute-angled prism, and under the most favourable circumstances for seeing the more refractive rays; nevertheless the spectrum was cut off so suddenly about the bundle of rays at G that I cannot be sure of the exact measurement, but think it differs little from the line γ of hydrogen.

Specimen.	Temp. C.	Sp. Gr.	μ_A.	μ_B.	μ_C.	μ_D.	μ_E.	μ_F.	μ_G.
I....	22°	1·460	1·5026	1·5076	1·5096	1·5180	1·5289		
II....	15·5	1·4705	1·5063	1·5117	1·5146	1·5224	1·5329		
II....	13·4	1·474	1·5071	...	...	1·5230	...	1·5446	1·5650

From these observations we obtain the following molecular refraction, $\frac{\mu-1}{d}P=R$.

Specimen.	R_A.	R_B.	R_C.	R_D.	R_E.	R_F.	R_G.
I.......	67·47	68·14	68·41	69·54	71·00		
II.......	67·48	68·20	68·59	69·63	71·03		
II.......	67·43	...	...	69·54	...	72·42	75·13

It is evident that in the iron compound also the refraction and dispersion are very great. The molecular refraction for the line *a* of hydrogen may be assumed at 68·5, and the molecular dispersion between the lines γ and α of hydrogen at 6·6. If we reckon for the lines A and H, we may assume H would be about 78·0, which would give a molecular dispersion between these limits of about 10·5.

In discussing their observations, Messrs. Mond and Nasini, assuming that CO has a molecular refraction of 8·4, reckoned the atomic refraction of nickel at 24·1, instead of 9·9, which I had assigned to this element in solutions of its salts, and which subsequently has been fairly corroborated by Nasini's own experiments. This difference they attributed to a difference of valency in the metal. By parity of reasoning the atomic refraction of iron for the line C would be 26·5, or 25·5 for the line A, instead of 11·6 previously determined for bivalent iron from solutions of its salts.

But this involves the assumption that CO in these carbonyls has the same value as we should expect to find in an organic substance. It also assumes that the nickel has 8 valencies, while the iron has 10, in the above-named compounds, and that iron has presumably $\frac{7}{2}$ in diferrohepta-carbonyl; and that potassium in its compounds with CO has still other valencies. There is also another difficulty; for if we determine the atomic dispersion of nickel and iron on the same principle, it will be found to be 4·8 and 5·2 respectively for the interval between the lines γ and α*; these numbers are about one fifth of the atomic refraction of the two metals given above, viz., 24·1 and 26·5. But the ratio between the atomic dispersion $(\gamma-\alpha)$ and the refraction of the most dispersive elements hitherto calculated, sulphur and phosphorus, is only about one tenth.

It seems more probable that the metals really retain their ordinary valency, and that the excessive refraction and dispersion is to be sought rather in the peculiar arrangement of the CO. In such compounds we may imagine the CO playing a part similar to that of the CH_2 in ordinary organic compounds, which may be increased or diminished in number without altering the general type of the substance. In fact I accept on optical grounds as well as chemical, the ring-formulæ indicated in Mond's lecture at the Royal Institution.

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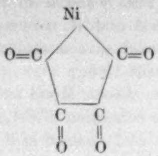
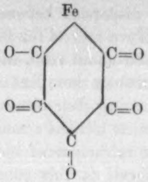
* Proc. R. S. xlii. p. 401.





In that case the molecular refraction due to each CO would be, from the nickel compound about 11·9, and from the iron compound about 11·3.

Although the atomic dispersion of nickel or of iron has not yet been definitely measured, it cannot greatly exceed 0·5 for $\gamma-a$. It is evident therefore from the figures for the dispersion, viz., $Ni(CO)_4=5\cdot9$, $Fe(CO)_5=6\cdot6$, that the molecular dispersion of each CO must be about the same in these two compounds, viz. 1·3, or thereabouts.

II. Indium and Gallium.

In 1885 I calculated the atomic refraction of indium and gallium from determinations of the refraction of certain alums made by M. Charles Soret. Very shortly afterwards he published another paper giving fresh and additional values for alums containing these metals *.

For indium the available data are derived from rubidium and cæsium indium alums, and for gallium from no fewer than five compounds, viz., potassium, rubidium, cæsium, thallium, and ammonium gallium alums. Calculated from these data, I find as mean values for the atomic refraction of these two metals the following :—

Metal.	Atomic Weight.	Specific Refraction.	R_A.
Indium	113·4	0·121	13·7
Gallium...............	69·9	0·166	11·6

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* *Archives Sc. Ph. & N. Genève*, xiv. p. 96.

These no doubt are nearer the truth then the higher figures previously given, though they must still be looked upon as only approximate. In regard to the atomic dispersion of these metals, the new observations quite confirm the previous remark that the order is "iron far the highest, chromium, indium, gallium, and aluminium lowest."

III. SULPHUR.

In the following Table is given the atomic refraction of sulphur, either uncombined, or in very simple combination:—

Condition.	A.	C, or "red."	D, or "white."	F.	G or γ.	H.
Solid	......	15·7	16·0			
Liquid	15·98	......	16·47			
Gaseous	......	16·1	......			
In solution	15·5	15·7	16·0	16·7	17·3	
From CS_2	15·8	16·05	16·3	17·05	17·7	18·4
,, Cl_2S	......	15·8	16·0			
,, Cl_2S_2	......	15·9	16·1			
,, Br_2S_2	......	16·35	......			

Sulphur in the solid condition is deduced from the observations of Descloiseaux and other physicists; but as the crystals of sulphur give three different indices of refraction for the same ray, the arithmetical mean of these three indices has been simply taken as the basis of calculation.

Sulphur in a liquid condition is from the observations of myself and the late Pelham Dale. It agrees, as far as D is concerned, with a determination by Becquerel. $R_{E}=16{\cdot}86$.

Sulphur in the gaseous condition was determined by Le Roux in 1861. The specific refraction was calculated out by me shortly afterwards, and recently by Nasini and Costa*. I give their number for the atomic refraction in the Table.

The sulphur in solution is from five determinations of this element dissolved in carbon bisulphide which I made some years ago with another object, and which have never been published. The solutions contained from about 22 to 27 per cent. of sulphur. Two determinations of somewhat weaker solutions were lately published by Nasini and Costa, and were practically identical with mine. The refraction of 17·3 for the line γ is deduced from their figures alone.

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* *Universita Inst. Ch. Roma*, 1891.

The four last lines give the values of sulphur reckoned from its simplest compounds.

In the case of bisulphide of carbon the figures in the Table are derived from my own most recent observations, which agree closely with those of other observers. The deduction made for the atom of carbon is 5·0 for the line A, 5·26 for the line H, and proportional numbers for the intermediate lines.

In the case of bichloride of sulphur the calculations have been made from the observations of Costa; and in that of chloride of sulphur from those of Haagen, Becquerel, and Costa, which fairly agree. The refraction due to an atom of chlorine is assumed to be 9·95 for the line C, and 10·05 for the line D. Costa, in treating of these substances, has adopted a slightly smaller value for chlorine.

In the case of the bromine compound the sulphur is calculated from Becquerel's observations, taking the value for bromine at 15·35.

The figures for sulphur derived from these various sources are very similar. I doubt, however, whether the sulphur dissolved in bisulphide of carbon and the sulphur which forms part of that compound do exert exactly the same influence upon the rays of light. If we were to reckon the value of carbon in bisulphide of carbon by deducting the value found for sulphur in solution, we should get for A, $36·6-31·0=5·6$, a higher figure than we ever find for carbon in a saturated compound. The difference would be still greater if reckoned by Lorenz's formula.

The accordance of the dispersion as exhibited throughout the Table is worthy of notice.

From the elaborate paper of Nasini and Costa, already referred to, it would appear that in some organic compounds, such as the xanthates, sulphur has at least the value of 16·0; but in the large majority of cases it has a distinctly lower value. In its oxygen compounds it is known to be far less refractive and dispersive.

IV. LIQUEFIED OXYGEN, NITROUS OXIDE, AND ETHYLENE.

In the recent paper of Professors Liveing and Dewar (Phil. Mag., Aug. 1892) they give determinations of the specific refraction of liquefied oxygen, nitrous oxide, and ethylene. It is, of course, possible to compare these interesting results with what theory would have led us to expect. The authors have themselves done so in the case of liquid oxygen at its boiling-point of −182° C. They remark that the refraction-equivalent, 3·182, which they found, differs but little from that deduced from gaseous oxygen at the ordinary temperature, viz. 3·0316, and "corresponds closely with the refraction-equivalent deduced by Landolt from the refractive indices of a number of organic compounds," which was 3·0. It actually comes between the two values, 2·8 and 3·4, which were assigned by Brühl to oxygen in its different states of combination with carbon, and to which an intermediate value has since been added.

Liveing and Dewar were able to determine the refraction of liquefied nitrous oxide for six different wave-lengths, the extremes being the red ray of lithium and G. These gave for $\frac{\mu-1}{d}$ 0·2595 and 0·2691 respectively, and for the molecular refraction of the red ray 11·418, and of G 11·840. The molecular dispersion between G and the lithium-line is therefore 0·422, which would indicate about 0·63 between H and A. Now it is difficult to say what the rational composition of nitrous oxide is, and therefore what its theoretical refraction and dispersion should be. Nitrogen in ammonia and its congeners is reckoned at 5·1, in nitriles at 4·1; oxygen double-bonded is reckoned at 3·4, with two separate bonds at 2·8. We therefore, according to our theoretical views, might reckon nitrous oxide at 13·6, 13·0, 11·6, or 11·0. The probability is therefore clearly in favour of the lower figure for nitrogen. As the dispersion-equivalent of nitrogen in ammonia is as high as 0·38, this would also seem to exclude the idea of the nitrogen in nitrous oxide being in the same condition as in ammonia. This will probably be considered the most likely alternative also on chemical grounds.

On calculating Messrs. Liveing and Dewar's numbers for liquid ethylene, we obtain the molecular refraction of 17·2 for the line D. But Professor Dewar has kindly furnished me with a more recent determination of what he believes to be a purer specimen. The refractive index, at the boiling-point, for the line C is 1·3445, and for the line F is 1·3528, the specific gravity being 0·55. These figures give 17·53 as the molecular refraction for the line C, and 17·92 for the line F. These will indicate about 17·41 for the line A. The theoretical value for ethylene, for the line A, is 17·4.

C_2 . . . . = 10·0
H_4 . . . . = 5·2
Double-linking . . . = 2·2
17·4

The coincidence between experiment and theory is very striking.

PROCEEDINGS

OF

THE PHYSICAL SOCIETY

OF LONDON.

JUNE 1893.

X. *On Plane and Spherical Sound-Waves of Finite Amplitude.* *By* CHARLES V. BURTON, *D.Sc.*[*]

PART I.—PLANE WAVES.

1. THE subject of plane waves of finite amplitude has been considered by Riemann[†]; and so long as we confine our attention to the case where velocity and density are everywhere continuous, his investigation, as is well known, leaves little to be desired. It will not, therefore, be necessary here to make further reference to this aspect of the subject; but there is one part of Riemann's work which Lord Rayleigh has clearly shown to be unsatisfactory, and it is this point which we have now especially to consider. Lord Rayleigh says[‡]:—

"..... It has been held that a state of motion is possible in which the fluid is divided into two parts by a surface of discontinuity propagating itself with constant velocity, all the fluid on one side of the surface of discontinuity being in one uniform condition as to density and velocity, and on eht other side in a second uniform condition in the same respects. Now, if this motion were possible, a motion of the same kind in which the surface of discontinuity is at rest would also be possible, as we may see by supposing a velocity equal and opposite to that with which the surface of discontinuity at first moves, to be impressed upon the whole mass of fluid. In order to find the relations which must subsist between the velocity and density on the one side (u_1, ρ_1) and the velocity and density on the other side (u_2, ρ_2), we notice in the first place that by the principle of the conservation of matter $\rho_2 u_2=\rho_1 u_1$. Again, if we consider the momentum of a slice bounded by parallel planes and including the surface of discontinuity, we see that the momentum leaving the slice in the unit of time is for each unit of area $(\rho_2 u_2=\rho_1 u_1)u_2$, while the momentum entering it is $\rho_1 u_1^2$. The difference of momentum must be balanced by the pressures acting at the boundaries of the slice, so that

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[*] Read February 24, 1893.

[†] "Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite," *Gött. Abhandl.* t. viii. (1860); reprinted in *Werke*, p. 145.

[‡] Theory of Sound, vol. ii. § 253, p. 41.

$$\rho_1 u_1(u_2-u_1)=p_1-p_2=a^2(\rho_1-\rho_2),$$

whence

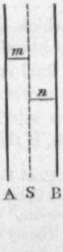
$$u_1=a\sqrt{\left(\frac{\rho_2}{\rho_1}\right)},\quad u_2=a\sqrt{\left(\frac{\rho_1}{\rho_2}\right)}.$$

The motion thus determined is, however, not possible; it satisfies indeed the conditions of mass and momentum, but it violates the condition of energy expressed by the equation

$$\tfrac{1}{2}u_2^2-\tfrac{1}{2}u_1^2=a^2\log\rho_1-a^2\log\rho_2."$$

2. The assumed motion here criticised is one in which density and velocity are constant for all points on the same side of the surface of discontinuity, while this surface itself is propagated through the fluid with constant velocity. It is easily shown, however, that the same objection applies when, on either side of the surface, velocity and density vary continuously in the direction of propagation, while the velocity of propagation of the surface is also allowed to vary. For let S (fig. 1) be a surface of discontinuity which is being propagated through the fluid, while the planes A, B, parallel to S and lying on either side of it, are fixed in the fluid. At a given instant let

Fig. 1.



distance of S from A $= m$,

" B " $S = n$;

density and velocity of fluid just to the left of $S = \rho_1, u_1$, density and velocity of fluid just to the right of $S = \rho_2, u_2$; velocity with which S is travelling $= V$.

Then, since A and B are fixed in the fluid, they are approximately moving with the respective velocities u_1, u_2; m and n being taken sufficiently small. On the same understanding, the mass of fluid between A and B (referred to unit surface) $= m\rho_1 + n\rho_2$; and since this mass must remain constant,

$$\frac{d}{dt}(m\rho_1 + n\rho_2) = 0;$$

i. e. in the limit, when m and n are infinitesimal,

$$\rho_1 \frac{dm}{dt} + \rho_2 \frac{dn}{dt} = 0,$$

or

$$\rho_1(V - u_1) = \rho_2(V - u_2). \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, if p_1 and p_2 are the pressures corresponding to ρ_1 and ρ_2, the principle of momentum gives:—

$$\begin{aligned} p_1 - p_2 &= \text{rate of change of momentum between A and B} \\ &= \frac{d}{dt}(u_1\rho_1 m + u_2\rho_2 n) \\ &= u_1\rho_1(V - u_1) - u_2\rho_2(V - u_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2) \end{aligned}$$

If the energy per unit volume corresponding to density ρ (in the absence of bodily motion) is called $\chi(\rho)$, the principle of energy would further give

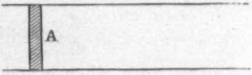
$$\begin{aligned} p_1u_1 - p_2u_2 &= \text{rate of change of energy between A and B} \\ &= \frac{d}{dt}\left\{m\left(\tfrac{1}{2}\rho_1u_1^2 + \chi(\rho_1)\right) + n\left(\tfrac{1}{2}\rho_2u_2^2 + \chi(\rho_2)\right)\right\} \\ &= \left\{\tfrac{1}{2}\rho_1u_1^2 + \chi(\rho_1)\right\}(V - u_1) - \left\{\tfrac{1}{2}\rho_2u_2^2 + \chi(\rho_2)\right\}(V - u_2). \quad (3) \end{aligned}$$

Since (1), (2), and (3) involve only the instantaneous values of u_1, ρ_1, u_2, ρ_2, and V, together with explicit functions of such values, while the space- and time-variations of all these quantities are absent from the equations, it is evident that the conditions to be satisfied at the surface S are the same as if

u_1, ρ_1, u_2, ρ_2, V were absolute constants. We conclude then, that, *with our assumptions*, a surface of discontinuity cannot be propagated through a fluid with any velocity, uniform or variable, except under that special law of pressure for which progressive waves are of accurately permanent type.

3. What, then, becomes of waves of finite amplitude after discontinuity has set in? We may emphasize this difficulty, and at the same time obtain a clue to its solution, by considering the following case (fig. 2):—A is a piston fitting a cylindrical tube (or, if we please, is a portion of an unlimited rigid plane). All the air to the right of A is initially at rest and of uniform density, and then A is impulsively set in motion, and kept moving to the right with uniform velocity v. Consider the speed with which the disturbance generated by A advances into the still air to the right; it is evident that in all cases the front of the disturbance must advance faster than A. Take, then, the case in which

Fig. 2.



$$v > a,$$

where a is the propagation-velocity of infinitesimal disturbances. Two alternatives present themselves :—

(i.) If velocity and density are always either constant or *continuously* variable in the direction of propagation, the rate of propagation at any point will, in accordance with known principles, be

$$\sqrt{\frac{dp}{d\rho}} + u,$$

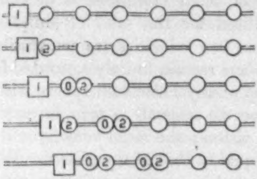
and therefore at the front of the disturbance, where $u = 0$ and $\rho =$ the "undisturbed" density, the velocity of propagation will be simply $= a$; that is, *less* than the velocity with which A is advancing. Obviously this will not do.

(ii.) If velocity and density are *not* always either constant or continuously variable, that is, if one or more surfaces of discontinuity are being propagated through the air, we are met by the difficulty explained in the last section.

4. A simple mechanical analogy will help to indicate the actual motion. A number of equal spheres, of the same

material throughout, are capable of sliding without friction along a straight bar (fig. 3), and are connected together by a number of very weak and exactly similar springs (not shown), so that when there is equilibrium they are equally spaced

Fig. 3.



along the bar. If one of the spheres were moved backwards and forwards through a small range, a disturbance would travel through the whole system, but owing to the weakness of the connecting springs it would travel very slowly. Suppose, now, that the last sphere on the left hand is connected to a movable piston by a spring half the length of the others, but otherwise similar to them; and let this piston be suddenly moved to the right with a considerable velocity which is kept constant, and which we may call unity. The weak connecting spring between the piston and the first sphere produces no sensible effect until the two are almost in contact, when the sphere rebounds with velocity 2. This first sphere then strikes the second, imparting to it the velocity 2, and at the same time coming to rest. The positions of the spheres after successive equal intervals of time are represented in fig. 3, where the number written on any sphere represents its velocity just *after* the impact which it is suffering. No number is written on those spheres which have not so far been affected by the motion. From this it will be evident that when the piston moves to the right with a constant velocity which is very great compared with the propagation-velocity of infinitesimal vibrations of the system, the disturbance advances to the right with twice the velocity of the piston, provided that the diameters of the spheres are excluded from the reckoning.

Now suppose that the spheres are too small and too close together to be individually distinguished; then, at any instant,

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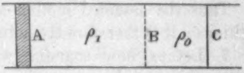
the system will appear to be divisible into two parts, in one of which the velocity is unity, while in the other it is zero; and in the moving part the spheres will appear to be twice as thickly condensed as in the still part. That the constant velocity of the piston is very great compared with the propagation-velocity of small vibrations is of course only a supposition introduced for the sake of simplicity. If, on the other hand, these two velocities are comparable, two adjacent spheres will always remain finitely separated from one another, and the velocity of any individual sphere within the disturbed stretch will never be as small as zero, or as great as twice the velocity of the piston; the *mean* velocity within the disturbed stretch being equal to that of the piston. When the spheres are very small and very close together, we shall still have apparently an abrupt transition from finite velocity and greater density to zero velocity and smaller density; and the energy, which is apparently lost as the spheres pass from the latter condition to the former, exists as energy of relative motion and unequal relative displacement amongst the spheres in the disturbed stretch.

5. Let us now compare the case just considered with the case of § 3 (fig. 2): and first, concerning the nature of the analogy, it should be noticed that the individual spheres are not the analogues of the separate gaseous molecules, but that when both spheres and molecules are very small and very numerous, the apparently continuous properties of the system of spheres correspond to similar properties of the gas. The connecting springs represent the elasticity of the gas, isothermal or adiabatic as the case may be, and the energy of relative motion and unequal relative displacement amongst the disturbed spheres *suggests* that there is a production of heat over and above that which would be due to the (isothermal or adiabatic) change of density; that is, *a dissipative production of heat*. The motion considered in the last section properly corresponds to the case where there is no conduction of heat, so that the connecting springs are the representatives of *adiabatic* elasticity, and the additional heat generated remains wholly within the more condensed part of the air. If we make the somewhat violent assumption that the temperature of the air remains constant throughout, the additional

heat generated will be conducted away isothermally, and the equivalent energy will be, for our purposes, entirely lost. To represent this case by means of our spheres we should have to regard the connecting springs as representing isothermal elasticity, while the energy of relative motion and unequal relative displacement among the disturbed spheres, as fast as it is produced, is to be consumed in doing work against suitable internal forces.

6. The mechanical system of spheres and springs, having suggested a solution, has served its purpose, and it now remains for us more closely to consider the aerial problem in the light of this suggestion. We may take, first, the case where the temperature is supposed to be invariable; for although such a supposition is necessarily far removed from the truth, it leads to very simple results, which indicate well enough the general character of the motion. Let the piston A (fig. 4) be moving to the right with constant velocity v (which may be either less or greater than a, the velocity of feeble sounds in air). Assume all the air between A and a parallel plane surface B to have the velocity v and density ρ_1, while all the air to the right of B is at rest and has the density ρ_0. Let the plane B move to the right with velocity V. Then the invariability of mass between A and a plane C fixed in the still air gives

Fig. 4.



$$\rho_1(\mathrm{V}-v)-\rho_0 v=0\,; \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

while from the principle of momentum,

$$\rho_1 v(\mathrm{V}-v)=p_1-p_0\,; \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

the pressure p being a function of ρ only, since the temperature is supposed to be constant throughout. If we assume for this case the truth of Boyle's law, so that $p=a^2\rho$ always, (5) becomes

$$\rho_1(a^2-\mathrm{V}v+v^2)=\rho_0 a^2, \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

which together with (4) is sufficient to determine V and ρ_1 when v and ρ_0 are given. Taking all these quantities to remain constant throughout the motion, we see that at each instant the following conditions are satisfied:—

(i) Every necessary condition between A and B, since density and velocity are there constant with respect to space and time;
(ii) Every necessary condition to the right of B, since the air there is at rest and in a constant uniform state;
(iii) Equality between the velocity of A and that of the air in contact with it;
(iv) At B, the conservation of mass and momentum, which are *necessary* conditions, and which, together with our supposition that the temperature is somehow maintained uniform, are *sufficient* to determine what takes place at B *.

Moreover, if at a time t (reckoned from the instant when A was impulsively started into motion) we take the distance of B from A to be $(V-v)t$, so that initially B coincides with A, the initial conditions are satisfied.

Thus the assumed motion satisfies all the necessary conditions; it is therefore the actual motion.

7. Let us now examine what occurs when no heat is allowed to pass by conduction or radiation; a state of things much more nearly realized in practice. Suppose the motion of A and the condition of the undisturbed air to be the same as in the last section, while the (constant) velocity of B is now called V', and the density and pressure of the air between A and B (called ρ', p' respectively) are also taken to be uniform and constant. At each instant, in place of (4) and (5), we shall now have

$$\rho'(V'-v)-\rho_0 v=0, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\rho' v(V'-v)=p'-p_0. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Since we assume that there is no transference of heat by conduction or radiation, the rate at which the *total* energy of the system increases must be equal to the rate at which work is being done upon it by the piston A. Let θ_0 be the absolute temperature to the right of B, that between A and B being θ', and let us further assume for simplicity that

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* Energy appears to be lost, because dissipatively produced heat is conducted away isothermally.

$$\frac{p}{\rho\theta} = \text{a const.};$$

while γ, the ratio of the two specific heats, is also supposed constant. It can then be shown without difficulty that the total energy *per unit mass* between A and B exceeds that to the right of B by

$$\frac{p_0}{(\gamma-1)\rho_0}\,\frac{\theta'-\theta_0}{\theta_0} + \frac{v^2}{2};$$

and multiplying this by $\rho_0 V'$, the mass of air which crosses one unit of the surface B in each unit of time, we obtain the rate (referred to unit area) at which the system is gaining energy. Again, the rate at which unit area of the piston does work on the system

$$= p'v = p_0 \frac{\rho'\theta'}{\rho_0\theta_0} \cdot v,$$

and equating this to the rate of gain of energy, we obtain

$$p_0 \frac{\rho'\theta'}{\rho_0\theta_0}\, v = \rho_0 V' \frac{v^2}{2} + \frac{p_0 V'}{(\gamma-1)\theta_0}(\theta'-\theta_0). \quad . \quad . \quad (9)$$

We may also write equation (8) in the form

$$\rho' v(V'-v) = \frac{p_0}{\rho_0\theta_0}(\rho'\theta' - \rho_0\theta_0); \quad . \quad . \quad . \quad (10)$$

and (7), (9), and (10) will then serve to determine V', ρ', θ' when v, ρ_0, θ_0 are given. Since we have taken all these quantities to remain constant throughout the motion, we see, as before, that at each instant all the necessary conditions are satisfied; the principles of mass and momentum, together with our supposition that there is no exchange of heat, being sufficient to determine what takes place at B. Again, if at a time t from the commencement of the motion we take the distance of B from A to be $(V'-v)t$, so that initially B coincides with A, the initial conditions are satisfied. The assumed motion thus satisfies all the necessary conditions, and is therefore the actual motion.

8. If we compare the results of the last two sections with

those given by Riemann*, we shall find complete accordance so far as § 6 is concerned, though with § 7 the case is different; and this may be easily explained. We cannot in general investigate the motion of a (frictionless) compressible fluid by means of the equations of continuity and momentum, without further making some supposition as to the exchange or non-exchange of heat, and so we usually assume either that the temperature remains constant, or that there is no exchange of heat: in either case (provided the motion is continuous), the pressure is a function of the density only. At a surface of discontinuity there is not only the ordinary heating effect due to compression, but also, as we have seen, a dissipative generation of heat, and so, when applying the equations of continuity and momentum at such a surface, we must know what becomes of this additional heat. Now in all cases Riemann makes the assumption that the pressure is a function of the density only, and this is necessarily equivalent to an assumption concerning the transference of heat. Throughout most of his treatment of waves of discontinuity Riemann assumes that temperature is constant and that Boyle's law holds good; accordingly our § 6 is entirely in harmony with his conclusions, in fact (4) and (5) are only particular forms of equations given by Riemann. Of course the hypothesis that a portion of gas can be instantaneously compressed to a finite extent without any appreciable change of temperature, is not in accordance with experience, *but provided we accept the assumption that the temperature remains constant throughout, all that Riemann says concerning the propagation of waves of discontinuity under Boyle's law will hold good.*

The assumption made in § 7, that there is no appreciable transference of heat, is probably much nearer the truth; but this is not in accordance with any assumption made by Riemann. When pressure is assumed to be a function of density only, and to vary with it according to the adiabatic law, *it is virtually assumed that at the discontinuity just so much heat remains in the gas as would be due to slow adiabatic compression, while the further amount of heat which is dissipa-*

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* *Loc. cit.*

tively produced is completely and instantaneously removed by conduction. But though Riemann's results may thus be justified by impossible assumptions concerning the diffusion of heat, we may more reasonably, following Lord Rayleigh, regard them as involving a destruction of energy. The real source of error lies in Riemann's fundamental hypothesis. At the outset he supposes the expansion and contraction of the air to be either purely isothermal or purely adiabatic, and thenceforward he treats the air as a frictionless and mathematically continuous fluid, in which pressure and density are connected by an invariable law. But in general the existence of such a fluid is contrary to the conservation of energy; for as soon as discontinuity arises, energy will be destroyed.

9. It may not be out of place to conclude this portion of the subject by a short reference to a paper by Dr. O. Tumlirz [*]. This author starts, as Riemann did, with the assumption that the pressure is a function of the density only, the law of pressure being further assumed to be the adiabatic law; and in order to avōid Riemann's error, he explicitly uses the principle of energy applicable to continuous motion, in place of the principle of momentum. But the foregoing discussion will have made it clear, I think, that the solution of the difficulty is not to be sought for in this direction. In addition to the assumptions common to his own work and to that of Tumlirz, Riemann uses only the principle of mass and the principle of momentum; and since by their aid alone he arrives at a completely determinate motion, it follows that any other motion consistent with the same arbitrary assumptions, and with the condition of mass, must violate the condition of momentum. We have seen, in fact, that there is dissipation of energy at a surface of discontinuity, so that the condition of energy applicable to continuous motion ceases to hold good. We are acquainted, too, with other instances where loss of continuity involves dissipation of energy; for example, there is the case of one hard body rolling over another.

As the result of his investigation, Dr. Tumlirz concludes

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[*] "Ueber die Fortpflanzung ebener Luftwellen endlicher Schwingungsweite," *Sitzungsb. der Wien. Akad.* xcv. pp. 367–387 (1887).

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that as soon as a discontinuity is formed it immediately disappears again, this effect being accompanied by a lengthening of the wave and a more rapid advance of the disturbance. In this way, therefore, he seeks to explain the increased velocity of very intense sounds, such as the sounds of electric sparks investigated by Mach[*]. But it has already been pointed out [§ 3 (i.)], that *when density and velocity are everywhere continuous functions of the coordinates*, the *front* of a disturbance advancing into still air must travel forward with the velocity of infinitely feeble sounds. A greater velocity can only ensue when the motion has become discontinuous.

PART II.—SPHERICAL WAVES.

10. When plane waves of finite amplitude are propagated through a frictionless compressible fluid, discontinuity must always occur sooner or later, and a moment's consideration will show that there are at least some cases when the motion in spherical waves becomes discontinuous; the question arises whether in any case it is possible (in the absence of viscosity) for divergent spherical waves to travel outward indefinitely without arriving at a discontinuous state. This question was suggested to me by Mr. Bryan, who at the same time kindly handed me notes of his manner of attacking the problem. His method was to write down the exact kinematical equation for spherical sound-waves, and then to obtain successive approximations to the integral of this equation. If it appears that after any number of approximations the integral would remain convergent for large values of the radius, we may conclude that our equation holds good throughout, and hence that no discontinuity arises. If, on the other hand, the second or any higher approximation becomes divergent for large values of the radius, it is probable that the motion becomes somewhere discontinuous. This method I have not followed out; but by another method which is, I hope, sufficiently conclusive, I shall now endeavour to show that discontinuity must always arise.

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[*] *Sitzungsb. der Wien. Akad.* lxxv., lxxvii., lxxviii. *Cf.* also W. W. Jacques ⌊On Sounds of Cannon⌉, Amer. Journ. Sci. 3rd ser. xvii. p. 116 (1879).

The case in which the motion loses its continuity comparatively early requires no further consideration here; we have only to concern ourselves with the case in which the initial disturbance has spread out into a spherical shell of very small disturbance whose mean radius is very great compared with the difference between its extreme radii. The equations applicable to the disturbance are then, very approximately,

$$u = a\frac{\delta\rho}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

$$u \text{ or } a\frac{\delta\rho}{\rho} \propto \frac{1}{r} \text{ for a given part of the wave,} \quad . \qquad (12)$$

where ρ is the mean density, $\rho + \delta\rho$ the actual density at a point where the velocity is u, and a is the velocity of infinitely feeble sounds in air of density ρ; r is as usual the distance of a point from the centre of symmetry. Let us consider two neighbouring points M and N, on the same radius, *each being fixed in a definite part of the wave*, the point M being *behind* N (*i.e.* nearer to the origin), and the air-velocity at M exceeding that at N by Δu. Then, as the wave advances, each part of it will be instantaneously moving forward with (very approximately) the velocity

$$\sqrt{\frac{dp}{d\rho}} + u,$$

determined by the corresponding values of ρ and u; so that M will be gaining on N at the rate

$$\Delta u + \frac{d}{d\rho}\sqrt{\frac{dp}{d\rho}} \cdot \frac{d\rho}{du}\Delta u \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

approximately. We may admit then that the rate at which M gains on N is

$$\text{never} < \mathrm{B}\Delta u,$$

where B is a constant suitably chosen.

Again, if $\Delta_0 u$ is the difference between the air-velocities at M and N at the time $t=0$, and r_0 is the corresponding coordinate of M, we may admit that

$$\Delta u \text{ is never} < \frac{\mathrm{A}r_0}{r_0 + at}\Delta_0 u,$$

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where A is a constant not very different from unity. Thus M gains on N at a rate which is

$$\text{never} < \mathrm{AB}\frac{r_0}{r_0+at}\Delta_0 u;$$

and between the times $t=0$ and $t=t_1$ the distance gained by M relatively to N will be

$$\text{at least } \mathrm{AB}\Delta_0 u\int_0^{t_1}\frac{r_0 dt}{r_0+at},$$

$$i.\,e. \qquad \text{at least } \mathrm{AB}\Delta_0 u\frac{r_0}{a}\log\frac{r_0+at_1}{r_0}. \quad . \quad . \quad . \quad (14)$$

If B is finite and positive this expression increases indefinitely with the time, so long as the laws of continuous motion hold good. If $\Delta_0 r$ was the distance between M and N at time $t=0$, the time required for M to *overtake*[*] N will be *not greater* than the value of t_1 given by

$$-\Delta_0 r=\mathrm{AB}\Delta_0 u\frac{r_0}{a}\log\frac{r_0+at_1}{r_0};$$

or, when M and N are taken indefinitely close together at starting, by

$$\log\frac{r_0+at_1}{r_0}=\frac{a}{r_0}\div\left\{\mathrm{AB}\left(-\frac{\partial u}{\partial r}\right)_0\right\};$$

$$i.\,e.\text{, we have } t_1 \leqq \frac{r_0}{a}\left(e^{\frac{a}{r_0}\div\left\{\mathrm{AB}\left(-\frac{\partial u}{\partial r}\right)_0\right\}}-1\right), \quad . \quad . \quad (15)$$

which gives us a finite upper limit to the time required for discontinuity to set in, provided B is finite. As our assumptions only remain approximately true so long as the motion is continuous, (15) will only give an approximation to the time when discontinuity first commences, and accordingly the relation must be taken to refer to that part of the wave for which its right-hand side is a minimum. If B is negative (which is not the case for any known substance), the appropriate part of the disturbance will be such that $\partial u/\partial r$ is positive.

To determine approximately the value of B, we may refer to (13) and the inequality immediately following. If we assume Boyle's law of pressure, so that $\surd(dp/d\rho)=\text{const.}$, we have evidently

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[*] *Cf.* Lord Rayleigh, 'Theory of Sound,' vol. ii. p. 36.

$$B=1 \text{ very nearly.}$$

If we assume that the changes of density take place adiabatically, so that $p \propto \rho^{\gamma}$ and γ is nearly constant, the approximate value of B becomes

$$1+\frac{d}{d\rho}\sqrt{\frac{dp}{d\rho}}\cdot\rho\div\sqrt{\frac{dp}{d\rho}}$$

by means of (11);

$$=\frac{\gamma+1}{2}.$$

If, then, viscosity be neglected, we must conclude that under any practically possible law of pressure the motion in spherical sound-waves always becomes discontinuous, and *à fortiori* the same will be true of cylindrical waves. But inasmuch as our result for spherical waves depends on the existence of an infinite *logarithm* in (14) when t_1 is increased without limit, we may conclude that for waves diverging in four dimensions (or, more generally, in any number of dimensions finitely greater than three) there would be some cases where the motion remained always continuous.

11. The general question of spherical sound-waves of finite amplitude is by no means an easy one. In the case of plane waves we can write down at once from Riemann's equations the condition that the disturbance may be propagated wholly in the positive or wholly in the negative direction. The respective conditions are[*]:—

$$u=\pm\int_{\rho_0}^{\rho}\sqrt{\frac{dp}{d\rho}}\cdot d\log\rho,$$

where ρ_0 is the density of that part of the fluid whose velocity is reckoned as zero. No such simple criterion can be given for the existence of a purely convergent or purely divergent spherical disturbance; a fact which may be readily seen from the equations for waves of infinitesimal amplitude. If ϕ is the potential of a purely divergent system of waves, we have

$$r\phi=f(at-r), \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

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[*] *Cf.* also Lord Rayleigh, 'Theory of Sound,' vol. ii. p. 35 (3).

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where f is a function whose form is unrestricted. Let ρ be the ordinary density of the air, and $\rho+\delta\rho$ the actual density at a point where the coordinate is r and velocity u. We have, then, on differentiating (16) the well-known relations

$$u=\frac{\partial\phi}{\partial r}=-\frac{f(at-r)}{r^2}-\frac{f'(at-r)}{r} \quad . \quad . \quad . \quad (17)$$

and

$$a\frac{\delta\rho}{\rho}=-\frac{1}{a}\frac{\partial\phi}{\partial t}=-\frac{f'(at-r)}{r}. \quad . \quad . \quad . \quad . \quad (18)$$

From (17) and (18),

$$\left(a\frac{\delta\rho}{\rho}-u\right)r^2=f(at-r),$$

whence differentiating with respect to r, and neglecting small quantities beyond the first order,

$$r^2\left(\frac{a}{\rho}\frac{\partial\rho}{\partial r}-\frac{\partial u}{\partial r}\right)+2r\left(a\frac{\delta\rho}{\rho}-u\right)=-f'(at-r)$$

$$=ar\frac{\delta\rho}{\rho}$$

by (18) ; therefore

$$\frac{a}{\rho}\frac{\partial\rho}{\partial r}-\frac{2u}{r}+\frac{a}{r}\frac{\delta\rho}{\rho}-\frac{\partial u}{\partial r}=0. \quad . \quad . \quad . \quad . \quad (19)$$

If, then, an infinitesimal spherical disturbance is to be purely divergent, this equation must be satisfied for every value of r. But since the left-hand side involves $\delta\rho/\rho$ as well as u, $\partial u/\partial r$, and $\partial(\log\rho)/\partial r$, it is evident that the question whether or not the equation is satisfied for some particular value of r does not depend solely on the state of things in the immediate neighbourhood of this value, but is influenced also by the value of ρ corresponding to the undisturbed air. We must not therefore seek to characterize a purely divergent disturbance by a differential equation expressing that, with respect to the air at each point, the disturbance is wholly propagated in the positive direction of r.

12. Not recognizing this, I had attempted to discover such an equation, and one step of the inquiry is reproduced here, for the sake of any interest which it may have.

It is required to write down the differential equation of an *infinitesimal spherical disturbance, which is superposed on a purely radial steady motion.*

Though a steady motion extending inward to the pole would involve a violation of the principle of continuity, we may suppose that throughout a shell of finite thickness the distribution of density and velocity is such as would be consistent with steady motion; the motion within such a shell would then continue steady, provided that its spherical boundaries were constrained to expand or contract in a suitable manner. In the absence of constraints the shell of steady motion would be invaded from without and from within by disturbances emanating from adjoining parts of the fluid, but, at points well within the shell, the character of the steady motion would necessarily be maintained for a finite time.

Let ϕ be the potential of the steady motion.

Let $\phi+\psi$ be the potential of the actual motion so that ψ and its derivatives are small.

Let p, ρ be the pressure and density in the steady motion.

Let $p+\delta p$, $\rho+\delta\rho$ be the pressure and density in the actual motion, and assume that the pressure is a function of the density only. From the ordinary equations for the motion of compressible fluids we obtain

$$\int^{p}\frac{dp}{\rho}=-\tfrac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

$$\begin{aligned}\int^{p+\delta p}\frac{dp}{\rho}&=-\tfrac{1}{2}\left(\frac{\partial\phi}{\partial r}+\frac{\partial\psi}{\partial r}\right)^2-\dot{\psi}\\ &=-\tfrac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2-\frac{\partial\phi}{\partial r}\frac{\partial\psi}{\partial r}-\dot{\psi}, \quad . \quad . \quad . \qquad (21)\end{aligned}$$

when small quantities of the second order are neglected. Subtracting (20) from (21),

$$\frac{\delta p}{\rho}=-\dot{\psi}-\frac{\partial\phi}{\partial r}\frac{\partial\psi}{\partial r}. \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

Now

$$\frac{\delta p}{\rho}=\frac{dp}{d\rho}\cdot\frac{\delta\rho}{\rho};$$

therefore

$$\frac{\partial}{\partial t}\frac{\delta p}{\rho} = \frac{dp}{d\rho}\cdot\frac{1}{\rho}\frac{\partial\,.\,\delta\rho}{\partial t}\,; \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

and the equations of continuity for the steady motion and the actual motion may be written

$$0 = \frac{\partial\rho}{\partial t} = -\rho\nabla^2\phi - \frac{\partial\phi}{\partial r}\frac{\partial\rho}{\partial r},$$

$$\frac{\partial(\rho+\delta\rho)}{\partial t} = -(\rho+\delta\rho)\nabla^2(\phi+\psi) - \left(\frac{\partial\phi}{\partial r}+\frac{\partial\psi}{\partial r}\right)\frac{\partial(\rho+\delta\rho)}{\partial r},$$

whence by subtraction

$$\frac{\partial\delta\rho}{\partial t} = -\rho\nabla^2\psi - \delta\rho\nabla^2\phi - \frac{\partial\rho}{\partial r}\frac{\partial\psi}{\partial r} - \frac{\partial\delta\rho}{\partial r}\frac{\partial\phi}{\partial r}. \quad . \qquad (24)$$

Again,

$$\frac{\partial\delta\rho}{\partial r} = \frac{\partial}{\partial r}\left\{\left(\frac{dp}{d\rho}\right)^{-1}\delta p\right\}$$

$$= \frac{\partial}{\partial r}\left\{\rho\left(\frac{dp}{d\rho}\right)^{-1}\left(-\dot\psi - \frac{\partial\phi}{\partial r}\frac{\partial\psi}{\partial r}\right)\right\};$$

expanding this and substituting in (24) we get

$$\frac{\partial\delta\rho}{\partial t} = -\rho\nabla^2\psi - \rho\left(\frac{dp}{d\rho}\right)^{-1}\left(-\dot\psi - \frac{\partial\phi}{\partial r}\frac{\partial\psi}{\partial r}\right)\nabla^2\phi - \frac{\partial\rho}{\partial r}\frac{\partial\psi}{\partial r}$$

$$- \left[\frac{\partial}{\partial r}\left\{\rho\left(\frac{dp}{d\rho}\right)^{-1}\right\}\left(-\dot\psi - \frac{\partial\phi}{\partial r}\frac{\partial\psi}{\partial r}\right)\right.$$

$$\left. + \rho\left(\frac{dp}{d\rho}\right)^{-1}\left(-\frac{\partial\dot\psi}{\partial r} - \frac{\partial\phi}{\partial r}\frac{\partial^2\psi}{\partial r^2} - \frac{\partial^2\phi}{\partial r^2}\frac{\partial\psi}{\partial r}\right)\right]\frac{\partial\phi}{\partial r}\,. \qquad (25)$$

Now differentiate (22) with respect to *t* and we have, remembering (23),

$$-\ddot\psi - \frac{\partial\phi}{\partial r}\frac{\partial\dot\psi}{\partial r} = \frac{1}{\rho}\frac{dp}{d\rho}\cdot\frac{\partial\delta\rho}{\partial t}.$$

In this equation we have to substitute the value of $\partial\delta\rho/\partial t$ from (25), and if we then put $\chi \equiv \psi r$, and perform the necessary reductions, we finally obtain as the differential equation satisfied by ψr,

$$\frac{\partial^2\chi}{\partial t^2} - (a^2-u^2)\frac{\partial^2\chi}{\partial r^2} + 2u\frac{\partial^2\chi}{\partial r\partial t} + \mathrm{U}\frac{\partial\chi}{\partial t} + \mathrm{V}\left(\frac{\partial\chi}{\partial r} - \frac{\chi}{r}\right) = 0,$$

where

$$a^2 \equiv \text{the variable } \frac{dp}{d\rho} \text{ (in the steady motion)},$$

$$u \equiv \frac{\partial\phi}{\partial r} \equiv \frac{\text{constant}}{\rho r^2},$$

$$\mathrm{U} \equiv \frac{1}{\rho}\frac{\partial(u\rho)}{\partial r} - \frac{2u}{a}\frac{\partial a}{\partial r} \equiv -\frac{2u}{r} - \frac{2u}{a}\cdot\frac{\partial a}{\partial r},$$

$$\mathrm{V} \equiv 2au\frac{\partial}{\partial r}\left(\frac{u}{a}\right) - \frac{a^2-u^2}{\rho}\frac{\partial\rho}{\partial r}.$$

If the steady motion in question is a state of rest, $u=0$ and ρ is a constant, so that $\mathrm{U}=0$, $\mathrm{V}=0$, and our equation reduces to the ordinary form for small spherical disturbances,

$$\frac{\partial^2\chi}{\partial t^2} - a^2\frac{\partial^2\chi}{\partial r^2} = 0.$$

If, on the other hand, $r=\infty$, the motion may, through any finite distance, be treated as linear. We shall then have u and ρ both constant, as well as a, and as before $\mathrm{U}=0$, $\mathrm{V}=0$. In that case

$$\frac{\partial^2\psi}{\partial t^2} - (a^2-u^2)\frac{\partial^2\psi}{\partial r^2} + 2u\frac{\partial^2\psi}{\partial r\partial t} = 0,$$

and this, by a change of independent variables, is easily seen to be the appropriate form for small plane disturbances of a fluid whose motion is otherwise uniform.

DISCUSSION.

Prof. A. S. Herschel enquired whether the nature of the solution for plane waves of finite amplitude was similar to that for ordinary wave-motion? In the latter case everything depended on the instantaneous impulses, for these alone determined the nature of the wave.

Referring to Mr. Bryan's paper, he (Prof. Herschel) asked if the author could apply his equations to centrifugal fans. A particular kind of double fan had recently been tested, and gave very anomalous results which had not yet been explained.

The President said Mr. Boys's experiments on flying-bullets might have some bearing on Dr. Burton's paper. If the conclusions there stated were correct, then the velocity of the air in front of a bullet should be greater than that of the bullet, even if the latter was travelling faster than ordinary

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sound-waves. He now asked Mr. Boys if his photographs gave any evidence of this.

Mr. Boys said the fact that the photographs showed disturbances in front of the bullet proved that the disturbance travelled faster. In one case, where a large bullet was moving at a velocity rather greater than that of ordinary sound in the medium, the front of the disturbance was about half an inch in advance of the bullet. In another instance, where the bullet was smaller and the velocity greater, the distance which the disturbance was in advance of the bullet was somewhat less. In all cases, even when the velocity of the bullet was four times that of sound, the character of the effects remained the same.

Dr. Burton replied to the points raised.

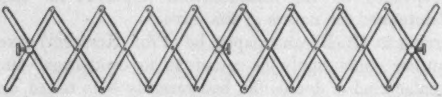
XI. *On a new and handy Focometer.*

By Professor J. D. Everett, *F.R.S.*[*]

This focometer is designed to permit the distance of the "object" from the screen to be varied, while the lens which is to throw on the screen an image of the "object" is automatically kept midway between the two. This position, as is well known, gives both the sharpest definition and the simplest calculation.

The instrument is constructed on the principle of the well-known toy called *lazy-tongs*. A number of flat bars (fig. 1), all exactly alike, are jointed together in such a way that half of them are in one plane and the other half in a superposed plane. With the exception of the end bars, each bar in either plane is jointed to three of the bars in the other plane, one

Fig. 1.



Plan.

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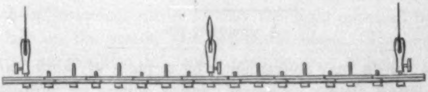
[*] Read February 24, 1893.

joint being in the middle and one at each end. The end bars are jointed at the middle and one end only. All the bars in the same plane are parallel, and the two sets together form a single row of rhombuses all equal and similar, a side of a rhombus being half the length of a bar. The system has only one degree of freedom, and its length is a definite multiple of the longitudinal diagonal of a rhombus.

The joints are arranged in three rows, one down the middle and one along each edge, and the distance from joint to joint in any row is equal to this longitudinal diagonal. This common distance can be varied between very wide limits by pulling out or pushing in the frame, and we have thus a means of dividing an arbitrary length into any number of equal parts. I utilize only the middle row for this purpose, and utilize it only or chiefly for bisecting a variable distance.

The pins on which the middle joints turn are continued upwards, as shown in fig. 2, to serve as supports for clips holding the object, the lens, and the screen. The lens is mounted on the centre pin, and the object and screen usually on the two end pins, as in fig. 2. In order to avoid flexibility, the clips are made short, and the pins, on which they are held by screws, rise only $1\frac{1}{2}$ inch above the frame. The base of each pin is a substantial disk (see fig. 3) which rests upon the table; all the pins, not only in the middle row, but also in the two outside rows, terminate in such disks, which serve as the feet of the instrument, and slide upon the table when the frame is expanded or contracted. The pins are of brass $\frac{1}{4}$ inch in diameter, and the bars are of $\frac{1}{4}$-inch mahogany, $\frac{3}{4}$ inch wide, and 13 inches in gross length. There are 18 of them, as shown in fig. 1. There are 9 pins in the middle row; and when the object and screen are on the two end pins, the distance between them is divided by the other pins into 8 equal parts, any two of

Fig. 2.



Elevation.

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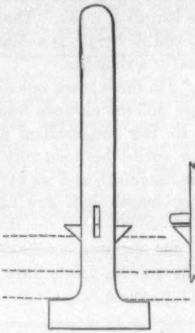
which should together make up the focal length. The unused pins are the most convenient handles for manipulating the frame.

Fig. 3.



A supporting pin.

The screen may conveniently be a piece of white card a little larger than a post-card, and a square of wire-gauze about half as big may be used as the object; but a still better 'object' is a cross of threads stretched across a square hole in a card. The light which passes through the square hole is very conspicuous on the screen before the correct distance is approached, whereas the shadow of the wire gauze is almost invisible. Two thin cards about the size of post-cards should be taken, a hole a centimetre square should be cut through both of them, and they should be gummed together with the cross threads between them, the threads being in the first instance long enough to project beyond the cards to facilitate adjustment while the gum is wet. Waxed carpet-thread, or any very stout thread with smooth edges, is the best for giving a conspicuous and at the same time a sharp image. As the cross will sometimes have to be raised or lowered, the hole should be much nearer to one end of the card than to the other, in order to give a greater range of adjustment in mounting on the clip. One thread should be vertical and the other horizontal, in order that their simultaneous focussing may serve as a check on the correct orientation of the lens.

The instrument is intended to be used by placing it on a table of length not less than four times the focal distance which is to be measured. A lamp is to be placed either on one end of the table or on a stand opposite the end, at such a height that its flame is about level with the tops of the clips. The clips should be fixed as low as possible on their supporting pins, unless it is necessary to raise them to suit the height of the lamp. In default of a lamp at the proper height, an adjustable mirror may be used instead, and made to reflect a beam of light from any large gas-flame in the room so that the beam shall pass along the tops of the clips. When the lamp or mirror has once been adjusted to throw its light in the proper direction, it should not be disturbed, as all necessary adjustments can be better made by moving the instrument.

The lens and screen may conveniently be mounted first, and the adjustments made so that the light collected by the lens falls on the screen as a horizontal beam. The cross is then to be mounted in such a position that a bright patch corresponding to the square hole is seen on the screen, surrounded by the shadow of the card. The frame must now be extended or compressed till the image of the cross appears in the bright patch; and the lens, object, and screen should then be carefully set square by hand before the final adjustment. If the vertical and horizontal lines of the cross do not focus simultaneously, it is a sign that the lens needs setting square.

The focussing having been completed, the distance of the object from the image is to be measured and divided by four. This will give the focal length; and the calculation can be checked by measuring one or more of the four equal parts into which the distance is divided by alternate pins. Owing to slight play in some of the joints, or other mechanical imperfections, the theoretically equal distances may exhibit sensible differences, especially when the frame is nearly closed up; but the method of observation is so well conditioned that these inequalities do not practically affect the correctness of the result.

In fact, if the distances of the lens from the object and image, instead of being exactly equal, are $a+x$ and $a-x$, the true focal length is $\frac{a^2-x^2}{2a}$, and in taking it to be one fourth of the whole distance we are simply neglecting x^2 in comparison with a^2. Suppose the two distances $a+x$ and $a-x$ to measure 20½ and 19½ inches, which is a larger inequality than is likely to occur, the ratio of x^2 to a^2 is 1 to 1600 ; and this error is negligible, in view of the fact that the doubt as to when the image is sharpest involves an uncertainty in the focal length to the extent usually of more than one per cent.

When the focal length does not exceed 10 or 12 inches, the instrument may be supported with the two hands and pointed towards a gas-flame, which need not be at the same level, but may be at any height. A fairly good measurement can thus be made by one person, if there is opportunity for setting the instrument down on a table or floor when the lens needs setting square, and when the final measurement of distance is to be made. The friction at the joints of the frame is just sufficient to keep them from working while the instrument is being carefully set down. The chief difficulty is from flexure.

Instead of receiving the image on a screen, it can be viewed in mid air. For this purpose I mount the cross on one of the two end clips, and a piece of wire gauze about the size of the palm of my hand on the other, setting the wires at a slope of 45° by way of contrast with the upright cross. The end which carries the cross should be turned towards the strongest light ; as this renders the cross more visible to an observer behind the gauze, and also renders the glistening wires of the gauze more visible when the observer stations himself behind the cross. The adjustment for focus is made by lengthening or shortening the frame till parallax is removed. This is a very convenient way of establishing experimentally the fact of the interchangeableness of object and image.

The instrument can also be employed to illustrate the general law of variation of conjugate focal distances, the lens being for this purpose shifted from the central pin to any one of the other pins, and the frame being then extended till the image is correctly focussed. Regarded as an optical bench, the instrument is remarkably light and handy. Its weight, including screen, cross, wire-gauze, and lens, is 2 lb. 10 oz.; and a lecturer can carry it through the streets of a town without inconvenience.

The dimensions and number of bars of the instrument as exhibited are recommended as the most convenient for general purposes. Ten bars only were constructed for the first trials, and any number included in the formula $4n+2$ might theoretically be employed.

In order to prevent looseness at the joints, it would be well to make the holes in the bars bear against a cone below and another cone above, with a very slightly tapering wedge for adjustment, as indicated in fig. 3.

If the instrument were to be set up permanently in one place, guides might be used for compelling the middle row of pins to travel without rotation, or the pin on which the lens is mounted might be a fixture; but as long as portability is to be preserved, I do not think that any arrangements for automatically preventing rotation would be practically beneficial. It is only in the large movements which precede the final adjustment that rotation occurs to any injurious extent.

The instrument has been constructed from my drawings by Messrs. Yeates of Dublin, and the cost is trifling.

DISCUSSION.

Mr. A. Hilger thought the instrument was too flexible to be used for accurate work.

Mr. Blakesley said it would be a great improvement if the pins could be prevented from rotating. For this purpose it might be advantageous to slot the heads of the pins so as to fit on a straight bar. He also suggested that by using a plane mirror close behind the lens the light would be reflected back and the length of the focometer could be reduced by one half.

The President thought Prof. Everett never intended the instrument to compete, as regards accuracy, with the elaborate and expensive apparatus now used; but nevertheless the focometer was a very valuable one, especially for students' work, and was particularly well adapted to impress upon them the facts relating to conjugate foci.

XII. *A Hydrodynamical Proof of the Equations of Motion of a Perforated Solid, with Applications to the Motion of a Fine Rigid Framework in Circulating Liquid.* By G. H. Bryan*.

Introduction.

1. In the whole range of hydrodynamics, there is probably no investigation which presents so many difficulties as that which deals with the equations of motion of a perforated solid in liquid. The object of the present paper is to show how these equations may be deduced directly from the pressure-equation of hydrodynamics, without having recourse to the laborious method of ignoration of coordinates. The possibility of doing this is mentioned by Prof. Lamb in his 'Treatise on the Motion of Fluids' (pp. 119, 120), but he dismisses the method with the brief remark that in most cases it would prove exceedingly tedious. I think, however, that it will be admitted that the following investigation is more straightforward and simple than that given by Basset in his 'Hydrodynamics,' vol. i. pp. 167–178.

The usual method presents little difficulty when the motion of the liquid is acyclic, because the whole motion could in such cases be set up from rest by suitable impulses applied to the solids alone; and a consideration of Routh's modified Lagrangian function shows that in this case the equations of motion can be obtained by expressing the total kinetic energy as a quadratic function of the velocity-components of the solid alone, and applying the generalized equations of motion referred to moving axes.

If, however, the solid is perforated, and the liquid is circulating through the perforations, this method presents several difficulties. If the solid were reduced to rest by the application of suitable impulses, the liquid would still continue to circulate through the perforations, the "circulation" in any circuit remaining unaltered. From this and other circumstances we are led to infer that these circulations are not generalized velocity-components, but rather that the quantities $\kappa\rho$ are generalized momenta. Now the kinetic energy

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* Read February 24, 1893.

of the system is naturally calculated as a function of the velocity-components of the solid and of these constant circulations (or the corresponding momenta); a form unsuited for obtaining the equations of motion. We ought either to have the kinetic energy expressed in terms of generalized velocity-components alone, or to know the "modified Lagrangian function" obtained by "ignoring" the velocity-components corresponding to the constant momenta or circulations. Either of these expressions involves constants which cannot be determined from the ordinary expression for the energy alone, and to determine them in the usual way it is necessary to resort to arguments based on a consideration of the "impulse" by which the motion might be set up from rest.

In the following investigation the equations of motion are deduced from purely hydrodynamical considerations, and from them the modified function is found. In §§ 12–16 the equations of motion are interpreted for the case in which the solid is a light rigid framework and the inertia is entirely due to the circulation of the liquid, and the results are applied to interpret the effective forces of the cyclic motion for a perforated solid in general.

General Hydrodynamical Equations.

2. Let a perforated solid bounded by the surface S be moving through an infinite mass of liquid (density ρ) with translational and rotational velocity-components u, v, w, p, q, r, referred to axes fixed in the solid, and let $\kappa_1, \kappa_2, \kappa_3 \ldots \kappa_m$ be the circulations in circuits drawn through the various apertures. Then we know that ϕ the velocity-potential of the fluid motion may be expressed as a linear function of the velocities and circulations in the form

$$\phi = u\phi_u + v\phi_v + w\phi_w + p\phi_p + q\phi_q + r\phi_r + \Sigma\kappa\phi_\kappa, \quad . \quad (1)$$

where evidently $\phi_u = \partial\phi/\partial u$ &c., and the coefficients $\phi_u \ldots$ depend only on the form of the solid.

If $d\nu$ denotes the element of the normal to S measured from the solid into the liquid, (l, m, n) its direction-cosines, then, in the usual way, we have

$$\frac{\partial\phi}{\partial\nu} = l(u - ry + qz) + m(v - pz + rx) + n(w - qx + py). \quad . \quad (2)$$

The six coefficients $\phi_u \dots \phi_r$ are single-valued functions of the coordinates, while the coefficients ϕ_κ which determine the part of the velocity-potential due to the circulations are cyclic functions making $\partial\phi_\kappa/\partial\nu=0$ at the surface of the solid; these coefficients are supposed known for each form of solid, although their determination in any given case is generally beyond the range of mathematical analysis.

3. Let $\sigma_1, \sigma_2, \dots \sigma_m$ be barriers drawn across the perforations; then, in the usual way, the kinetic energy of the liquid is found to be $\mathfrak{T}$, where

$$\mathfrak{T} = -\tfrac{1}{2}\rho\iint\phi\frac{\partial\phi}{\partial\nu}\,d\mathrm{S} + \tfrac{1}{2}\rho\Sigma\kappa\iint\frac{\partial\phi}{\partial\nu}\,d\sigma = \mathfrak{T}_1 + \mathrm{K}. \quad . \quad (3)$$

Here $\mathfrak{T}_1$ is a quadratic function of the velocity-components of the solid, and is the kinetic energy when the motion is acyclic, and K is a quadratic function of the circulations.

If the axes were fixed in space, the pressure equation (supposing no forces to act on the liquid) would be

$$\frac{p_1}{\rho} + \frac{\partial\phi}{\partial t} + \tfrac{1}{2}q_1{}^2 = \text{const.},$$

(where p_1 = pressure, q_1 = resultant velocity of liquid). Owing to the motion of the axes, however, $\partial\phi/\partial t$ must be replaced by the rate of change of ϕ at a fixed point, that is by

$$\frac{d\phi}{dt} - (u - yr + zq)\frac{\partial\phi}{\partial x} - (v - zp + xr)\frac{\partial\phi}{\partial y} - (w - xq + yp)\frac{\partial\phi}{\partial z},$$

whence the pressure equation becomes

$$\frac{p_1}{\rho} + \frac{d\phi}{dt} - (u - yr + zq)\frac{\partial\phi}{\partial x} - (v - zp + xr)\frac{\partial\phi}{\partial y} - (w - xq + yp)\frac{\partial\phi}{\partial z}$$
$$+ \tfrac{1}{2}q_1{}^2 = \text{const.} \quad . \quad . \quad . \quad . \quad (4)$$

The Mutual Reactions between the Solid and Liquid.

4. Let $\mathrm{X}_1, \mathrm{Y}_1, \mathrm{Z}_1, \mathrm{L}_1, \mathrm{M}_1, \mathrm{N}_1$ be the component forces and couples which the solid exerts on the liquid; then we have evidently

$$\mathrm{X}_1 = \iint l\,p_1\,d\mathrm{S}, \quad \mathrm{L}_1 = \iint(ny - mz)\,p_1\,d\mathrm{S}. \quad . \quad . \quad (5)$$

To reduce these expressions to the required form, we shall have to resort to repeated applications of Green's formula. Since the velocity-potential ϕ is a multiple-valued function, it follows that in transforming volume integrals involving ϕ we shall obtain surface integrals over the barriers $\sigma_1, \sigma_2, \ldots \sigma_m$ as well as over S the surface of the solid. On the other hand, the pressure p_1 and the velocity-components $\partial\phi/\partial x$, $\partial\phi/\partial y$, $\partial\phi/\partial z$ are single-valued and do not contribute barrier terms to the surface integrals. Moreover, since the circulations κ are independent of the time,

$$\frac{\partial\dot{\phi}}{\partial t} = u\phi_u + \dot{v}\phi_v + \dot{w}\phi_w + \dot{p}\phi_p + \dot{q}\phi_q + \dot{r}\phi_r;$$

and $\partial\phi/\partial t$ is therefore a single-valued function of the velocity-components of the solid satisfying Laplace's equation.

We also notice that

$$l = \frac{\partial\phi_u}{\partial\nu}, \quad ny - mz = \frac{\partial\phi_p}{\partial\nu}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

as may be at once seen by differentiating (2) with respect to u and p respectively.

5. Substituting for p_1 in (5) in terms of the velocities, we have

$$\begin{aligned}\frac{X}{\rho} = & -\iint l\frac{d\phi}{dt}\,dS \\ & + \iint\left\{(u - yr + zq)\frac{\partial\phi}{\partial x} + \text{(two similar)}\right\} l\,dS \\ & - \tfrac{1}{2}\iint\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\} l\,dS. \quad . \quad . \quad . \quad . \quad (7)\end{aligned}$$

The first line of this expression is, from (6), equal to

$$\begin{aligned}& -\iint\frac{d\phi}{dt}\frac{\partial\phi_u}{\partial\nu}\,dS \\ & = \iiint\left\{\frac{\partial\dot{\phi}}{\partial x}\frac{\partial\phi_u}{\partial x} + \frac{\partial\dot{\phi}}{\partial y}\frac{\partial\phi_u}{\partial y} + \frac{\partial\dot{\phi}}{\partial z}\frac{\partial\phi_u}{\partial z}\right\} dx\,dy\,dz\end{aligned}$$

by Green's transformation. Remembering that ϕ_u is independent of the time, this integral, taken throughout the liquid, becomes

$$\begin{aligned}& = \frac{d}{dt}\frac{\partial}{\partial u}\,\tfrac{1}{2}\iiint\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\} dx\,dy\,dz \\ & = \frac{1}{\rho}\frac{d}{dt}\frac{\partial\mathfrak{T}}{\partial u} = \frac{1}{\rho}\frac{\partial}{\partial t}\frac{\partial\mathfrak{T}_1}{\partial u}.\end{aligned}$$

By Green's transformation the second line is equal to

$$-\iiint \frac{\partial}{\partial x}\left\{(u-yr+zq)\frac{\partial \phi}{\partial x} + \text{(two similar)}\right\} dx\, dy\, dz$$

$$= -\iiint\left(r\frac{\partial \phi}{\partial y} - q\frac{\partial \phi}{\partial z}\right) dx\, dy\, dz$$

$$-\iiint\left\{(u-yr+zq)\frac{\partial}{\partial x} + \text{(two similar)}\right\}\frac{\partial \phi}{\partial x} dx\, dy\, dz,$$

which by a second application of Green's transformation becomes

$$=\iint(mr-nq)\phi dS - \Sigma\kappa\iint(mr-nq)d\sigma$$

$$+\iint\{l(u-yr+zq) + m(v-zp+xr) + n(w-xq+yp)\}\frac{\partial \phi}{\partial x}dS$$

$$=\iint(mr-nq)\phi dS - \Sigma\kappa\iint(mr-nq)d\sigma + \iint\frac{\partial \phi}{\partial \nu}\frac{\partial \phi}{\partial x}dS$$

by (2).

Lastly, the third line of (7) is, by Green's transformation,

$$=\iiint\left\{\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial \phi}{\partial z}\frac{\partial^2 \phi}{\partial x \partial z}\right\} dx\, dy\, dz$$

$$= -\iint\frac{\partial \phi}{\partial \nu}\frac{\partial \phi}{\partial x}dS.$$

Hence, adding the several terms together, we have

$$X_1 = \frac{d}{dt}\frac{\partial \mathfrak{T}}{\partial u}$$

$$+ r\{\rho\iint m\phi dS - \Sigma\kappa\rho\iint m d\sigma\} - q\{\rho\iint n\phi dS - \Sigma\kappa\rho\iint n d\sigma\} \quad . \quad (8)$$

Now by (6),

$$\rho\iint m\phi dS = \rho\iint\frac{\partial \phi_v}{\partial \nu}\phi dS$$

$$= \Sigma\kappa\rho\iint\frac{\partial \phi_v}{\partial \nu}d\sigma - \rho\iiint\left\{\frac{\partial \phi}{\partial x}\frac{\partial \phi_v}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \phi_v}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial \phi_v}{\partial z}\right\} dx\, dy\, dz$$

$$= \Sigma\kappa\rho\iint\frac{\partial \phi_v}{\partial \nu}d\sigma - \frac{\partial}{\partial v}\tfrac{1}{2}\rho\iiint\left\{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right\} dx\, dy\, dz$$

$$= \Sigma\kappa\rho\iint\frac{\partial \phi_v}{\partial \nu}d\sigma - \frac{\partial \mathfrak{T}}{dv}.$$

Therefore

$$X_1 = \frac{d}{dt}\frac{\partial \mathfrak{T}}{\partial u} - r\left\{\frac{\partial \mathfrak{T}}{\partial v} + \Sigma\kappa\rho\iint\left(m - \frac{\partial \phi_v}{\partial \nu}\right)d\sigma\right\}$$

$$+ q\left\{\frac{\partial \mathfrak{T}}{\partial w} + \Sigma\kappa\rho\iint\left(n - \frac{\partial \phi_w}{\partial \nu}\right)d\sigma\right\}. \quad . \quad . \qquad (9)$$

6. In like manner we have

$$\frac{L_1}{\rho} = -\iint (ny - mz)\frac{d\phi}{dt}\,dS$$

$$+ \iint\left\{(u - yr + zq)\frac{\partial \phi}{\partial x} + (\text{two similar})\right\}(ny - mz)\,dS$$

$$-\tfrac{1}{2}\iint\left\{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right\}(ny - mz)\,dS$$

$$= -\iint \frac{d\phi}{dt}\frac{\partial \phi_p}{\partial \nu}\,dS$$

$$- \iiint\left\{y\left(q\frac{\partial \phi}{\partial x} - p\frac{\partial \phi}{\partial y}\right) - z\left(p\frac{\partial \phi}{\partial z} - r\frac{\partial \phi}{\partial x}\right)\right\}dx\,dy\,dz$$

$$- \iiint\left[y\left\{(u - yr + zq)\frac{\partial}{\partial x} + (\text{two similar})\right\}\frac{\partial \phi}{\partial z}\right.$$

$$\left. - z\left\{(u - yr + zq)\frac{\partial}{\partial x} + (\text{two similar})\right\}\frac{\partial \phi}{\partial y}\right]dx\,dy\,dz$$

$$+ \iiint\left(\frac{\partial \phi}{\partial x}\frac{\partial}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial z}\right)\left(y\frac{\partial \phi}{\partial z} - z\frac{\partial \phi}{\partial y}\right)dx\,dy\,dz$$

$$= \frac{1}{\rho}\frac{d}{dt}\frac{\partial \mathfrak{T}}{\partial p}$$

$$+ \iint\left\{l(u - yr + zq) + (\text{two sim.}) - \frac{\partial \phi}{\partial \nu}\right\}\left(y\frac{\partial \phi}{\partial z} - z\frac{\partial \phi}{\partial y}\right)dS$$

$$- \iiint\left[y\left(q\frac{\partial \phi}{\partial x} - p\frac{\partial \phi}{\partial y}\right) - z\left(p\frac{\partial \phi}{\partial z} - r\frac{\partial \phi}{\partial x}\right)\right.$$

$$\left. - (v - zp + xr)\frac{\partial \phi}{\partial z} + (w - xq + yp)\frac{\partial \phi}{\partial y}\right]dx\,dy\,dz.$$

Remembering that in this expression one factor of the surface integral is zero at every point of S, we have, by again applying Green's transformation to the volume integral,

$$
\begin{aligned}
\mathrm{L}_1 = {} & \frac{d}{dt}\frac{\partial\mathfrak{T}}{\partial p} \\
& + w\{\rho\iint m\phi d\mathrm{S} - \Sigma\kappa\rho\iint m d\sigma\} - v\{\rho\iint n\phi d\mathrm{S} - \Sigma\kappa\rho\iint n d\sigma\} \\
& + r\{\rho\iint(lz - nx)\phi d\mathrm{S} - \Sigma\kappa\rho\iint(lz - nx)d\sigma\} \\
& - q\{\rho\iint(mx - ly)\phi d\mathrm{S} - \Sigma\kappa\rho\iint(mx - ly)d\sigma\} \quad . \quad . \quad (10)
\end{aligned}
$$

Now, just as before,

$$
\begin{aligned}
\rho\iint(lz - mx)\phi d\mathrm{S} & = \rho\iint\frac{\partial\phi_q}{\partial\nu}\phi d\mathrm{S} \\
& = \Sigma\kappa\rho\iint\frac{\partial\phi_q}{\partial\nu}d\sigma \\
& \qquad - \rho\iiint\left(\frac{\partial\phi}{\partial x}\frac{\partial\phi_q}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\phi_q}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial\phi_q}{\partial z}\right)dx\,dy\,dz \\
& = \Sigma\kappa\rho\iint\frac{\partial\phi_q}{\partial\nu}d\sigma - \frac{\partial\mathfrak{T}}{\partial q}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{L}_1 = {} & \frac{d}{dt}\frac{\partial\mathfrak{T}}{\partial p} \\
& - w\left\{\frac{\partial\mathfrak{T}}{\partial v} + \Sigma\kappa\rho\iint\left(m - \frac{\partial\phi_v}{\partial\nu}\right)d\sigma\right\} \\
& + v\left\{\frac{\partial\mathfrak{T}}{\partial w} + \Sigma\kappa\rho\iint\left(n - \frac{\partial\phi_w}{\partial\nu}\right)d\sigma\right\} \\
& - r\left\{\frac{\partial\mathfrak{T}}{\partial q} + \Sigma\kappa\rho\iint\left(lz - nx - \frac{\partial\phi_q}{\partial\nu}\right)d\sigma\right\} \\
& + q\left\{\frac{\partial\mathfrak{T}}{\partial r} + \Sigma\kappa\rho\iint\left(mx - ly - \frac{\partial\phi_r}{\partial\nu}\right)d\sigma\right\}. \quad . \quad (11)
\end{aligned}
$$

Application to the Equations of Motion.

7. The equations of motion of the solid may now be written down at once. Let $\mathfrak{T}'$ be the kinetic energy of the solid, T the total kinetic energy $= \mathfrak{T} + \mathfrak{T}'$; and suppose that the motion takes place under the action of a system of external impressed forces and couples designated by X, Y, Z, L, M, N. Then the effective forces and couples to which the motion of the solid itself is due are $\mathrm{X} - \mathrm{X}_1 \ldots$, $\mathrm{L} - \mathrm{L}_1 \ldots$, respectively, and

the six equations of motion of the solid referred to the moving axes are of the form

$$\frac{d}{dt}\frac{\partial\mathfrak{T}'}{\partial u}-r\frac{\partial\mathfrak{T}'}{\partial v}+q\frac{\partial\mathfrak{T}'}{\partial w}=\mathrm{X}-\mathrm{X}_1, \quad . \quad . \quad . \quad (12)$$

$$\frac{d}{dt}\frac{\partial\mathfrak{T}'}{\partial p}-w\frac{\partial\mathfrak{T}'}{\partial v}+v\frac{\partial\mathfrak{T}'}{\partial w}-r\frac{\partial\mathfrak{T}'}{\partial q}+q\frac{\partial\mathfrak{T}'}{\partial r}=\mathrm{L}-\mathrm{L}_1. \quad . \quad (13)$$

Hence, on substitution, we see that the required equations of motion are found by writing T for $\mathfrak{T}$ and X, Y, Z, L, M, N for X_1, Y_1, Z_1, L_1, M_1, N_1 in equations (9) (11). The resulting equations may be written :

$$\mathrm{X}=\frac{d}{dt}\frac{\partial \mathrm{T}}{\partial u}-r\left(\frac{\partial \mathrm{T}}{\partial v}+\eta\right)+q\left(\frac{\partial \mathrm{T}}{\partial w}+\zeta\right), \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\mathrm{L}=\frac{d}{dt}\frac{\partial \mathrm{T}}{\partial p}-w\left(\frac{\partial \mathrm{T}}{\partial v}+\eta\right)+v\left(\frac{\partial \mathrm{T}}{\partial w}+\zeta\right)$$
$$-r\left(\frac{\partial \mathrm{T}}{\partial q}+\mu\right)+q\left(\frac{\partial \mathrm{T}}{\partial r}+\nu\right), \quad (15)$$

where ξ, η, ζ, λ, μ, ν are defined by the equations

$$\xi=\Sigma\kappa\rho\iint\left(l-\frac{\partial\phi_u}{\partial\nu}\right)d\sigma, \text{ \&c.} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\lambda=\Sigma\kappa\rho\iint\left(ny-mz-\frac{\partial\phi_p}{\partial\nu}\right)d\sigma, \text{ \&c.} \quad . \quad . \quad . \quad (17)$$

As Lamb has pointed out ('Motion of Fluids,' p. 140), the six quantities (ξ, η, ζ, λ, μ, ν) are "the components of the impulse of the cyclic fluid motion which remains when the solid is (by forces applied to it alone) brought to rest"*. They are linear functions of the circulations and their form depends on the form of the solid. If there is only one aperture they are all proportional to the circulation κ.

The Modified Lagrangian Function.

8. We shall now show that the motion of the solid can be determined in terms of Routh's modified Lagrangian function,

---

* Our ξ, η, ζ, λ, μ, ν are the same as the ξ_0, η_0, ζ_0, λ_0, μ_0, ν_0 of Lamb, or the $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ of Basset's 'Hydrodynamics.'

and shall find the form of this function for the system. Putting

$$\mathrm{H}=\mathrm{T}+\xi u+\eta v+\zeta w+\lambda p+\mu q+\nu r+\mathrm{F}(\kappa\rho), \quad . \quad . \quad (18)$$

where $\mathrm{F}(\kappa\rho)$ denotes any function whatever of the quantities $\kappa\rho$, we see that the equations of motion reduce to the standard form

$$\mathrm{X}=\frac{d}{dt}\frac{\partial\mathrm{H}}{\partial u}-r\frac{\partial\mathrm{H}}{\partial v}+q\frac{\partial\mathrm{H}}{\partial w}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\mathrm{L}=\frac{d}{dt}\frac{\partial\mathrm{H}}{\partial p}-w\frac{\partial\mathrm{H}}{\partial v}+v\frac{\partial\mathrm{H}}{\partial w}-r\frac{\partial\mathrm{H}}{\partial q}+q\frac{\partial\mathrm{H}}{\partial r}. \quad . \quad (20)$$

The function H, therefore, plays the same part in determining the equations of motion of the solid as the kinetic energy T in the case of an imperforated solid (or any solid when the motion of the liquid is acyclic). It remains (i.) to determine what quantities are to be regarded as the generalized velocities if the quantities $\kappa\rho$ are regarded as generalized momenta ; (ii.) to find the form of the function $\mathrm{F}(\kappa\rho)$ in order that H may represent the modified Lagrangian function.

9. Let $\dot{\chi}_m$ be the generalized velocity-component corresponding to the ignored momentum $\kappa_m\rho$. Then, as Routh has shown ('Rigid Dynamics,' vol. i. § 420), the modified Lagrangian function H is of the form

$$\mathrm{H}=\mathrm{T}-\Sigma\kappa_m\rho\dot{\chi}_m, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and therefore by equating the two expressions for H we must have

$$\xi u+\eta v+\zeta w+\lambda p+\mu q+\nu r+\mathrm{F}(\kappa)=-\Sigma\kappa\rho\dot{\chi}. \quad . \quad . \quad (22)$$

Since $\dot{\chi}_m$ is the generalized velocity-component corresponding to the momentum $\kappa_m\rho$, therefore

$$\frac{\partial\mathrm{H}}{\partial\,.\,\kappa_m\rho}=-\dot{\chi}_m. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Now H is a homogeneous quadratic function of the six velocities $(u\ldots,p\ldots)$ and the momenta $\kappa\rho$; therefore

$$2\mathrm{H}=\Sigma u\frac{\partial\mathrm{H}}{\partial u}+\Sigma\kappa\rho\frac{\partial\mathrm{H}}{\partial\,.\,\kappa\rho}=\Sigma u\frac{\partial\mathrm{H}}{\partial u}-\Sigma\kappa\rho\dot{\chi}. \quad . \quad (24)$$

Hence, from (21),

$$2T=\Sigma u\frac{\partial H}{\partial u}+\Sigma\kappa\rho\dot{\chi}=\Sigma u\frac{\partial H}{\partial u}-\Sigma\kappa\rho\frac{\partial H}{\partial\,.\,\kappa\rho}. \quad . \quad . \quad (25)$$

The portions of T and H which involve only the momenta $\kappa\rho$, and are independent of the six velocities $(u\ldots, p\ldots)$, must arise from the terms $\Sigma\kappa\rho\chi$ in the above expressions (24) (25), and must therefore be equal and of opposite sign in the expressions T and H respectively. Hence, since from (3)

$$T=\mathfrak{T}'+\mathfrak{T}_1+K,$$

the portion of H which is independent of the six velocities $(u\ldots, p\ldots)$ must be $-K$, so that

$$\begin{aligned} H&=\mathfrak{T}'+\mathfrak{T}_1+(\xi u+\eta v+\zeta w+\lambda p+\mu q+\nu r)-K,\\ &=T\ +(\xi u+\eta v+\zeta w+\lambda p+\mu q+\nu r)-2K, \quad . \quad . \quad (26)\end{aligned}$$

and therefore $F(\kappa\rho)=-2K$. . . . . . (27)

The function $F(\kappa\rho)$ does not enter into the six equations of motion of the solid, but its form requires to be determined if we wish to reduce the equations of motion of the whole system to the canonical or Hamiltonian form.

The Generalized Velocities and Momenta.

10. Comparing (21) with (27), we see that

$$\Sigma\kappa\rho\dot{\chi}=2K-(\xi u+\eta v+\zeta w+\lambda p+\mu q+\nu r). \quad . \quad (28)$$

Now equation (3) may be written in the form

$$\begin{aligned}2(\mathfrak{T}_1+K)=&-\rho\iint\phi\frac{\partial}{\partial\nu}(u\phi_u+\ldots+p\phi_p+\ldots+\Sigma\kappa\phi_\kappa)dS\\ &+\Sigma\kappa\rho\iint\frac{\partial}{\partial\nu}(u\phi_u+\ \ldots+p\phi_p+\ldots+\Sigma\kappa\phi_\kappa)d\sigma. \quad (29)\end{aligned}$$

But by § 2, $\partial\phi_\kappa/\partial\nu=0$ all over the surface S of the solid. Hence, equating the terms independent of the six velocities $(u\ldots, p\ldots)$ on the two sides of (29), we have

$$2K=\Sigma\kappa\rho\iint\frac{\partial}{\partial\nu}(\Sigma\kappa\phi_\kappa)d\sigma. \quad . \quad . \quad . \quad . \quad (30)$$

But by (16) (17),

$$
\begin{aligned}
\xi u+\eta v+\zeta w+\lambda p+\mu q+\nu r \\
&= -\Sigma\kappa\rho\iint\frac{\partial}{\partial\nu}(u\phi_u+\ldots+p\phi_p+\ldots)\,d\sigma \\
&\quad +\Sigma\kappa\rho\iint\{lu+mv+nw+(ny-mz)p+\ldots\}\,d\sigma.
\end{aligned}
$$

Hence, by (28) and (30),

$$
\begin{aligned}
\Sigma\kappa\rho\dot{\chi} &= \Sigma\kappa\rho\iint\frac{\partial}{\partial\nu}(u\phi_u+\ldots+p\phi_p+\ldots+\Sigma\kappa\phi_\kappa)\,d\sigma \\
&\quad -\Sigma\kappa\rho\iint\{l(u-yr+zq)+m(v-zp+xr)+n(w-xq+yp)\}\,d\sigma;
\end{aligned}
$$

and therefore

$$
\dot{\chi}_m=\iint\left(\frac{\partial\phi}{\partial\nu}-l(u-yr+zq)-(\text{two similar})\right)d\sigma_m. \quad . \quad (31)
$$

Now $\partial\phi/\partial\nu$ is the velocity of the liquid resolved along the normal to the barrier σ_m, and

$$
l(u-yr+zq)+m(v-zp+xr)+n(w-xq+yp)
$$

is the velocity of the barrier σ_m resolved normally to itself at the point x, y, z, supposing the barrier to be fixed relatively to the solid. Their difference, therefore, represents the normal relative velocity of the liquid with respect to the barrier.

Hence $\dot{\chi}_m$ represents the total rate of flow of the liquid across the barrier σ_m relative to the solid; in other words, the generalized velocity corresponding to the ignored momentum $\rho\kappa_m$ is the volume of liquid per unit time flowing through the aperture relatively to the solid.

This property is proved in a different way by Basset in his 'Hydrodynamics,' vol. i. page 176.

The Form of the Modified Function.

11. It may be interesting to examine a little more closely the effect of the circulations on the motion of a solid.

When the motion of the liquid is acyclic, the kinetic energy is a homogeneous quadratic function of (u, v, w, p, q, r). In general it therefore involves 21 constants, but by a suitable choice of axes it is *always* possible to reduce this number by six, and a further reduction may be effected when the body is symmetrical. When the motion is cyclic the kinetic energy must be replaced by the modified function which in addition contains the seven terms

$$\xi u+\eta v+\zeta w+\lambda p+\mu q+\nu r-\mathrm{K},$$

of which the last term does not enter into the equations of motion of the solid. The six coefficients $(\xi, \ldots \lambda, \ldots)$ are linear functions of the circulations, and they remain constant so long as only conservative forces act on the liquid, for the circulations themselves then remain constant. Hence the modified function H may be regarded as a non-homogeneous quadratic function of the six velocities involving 28 constants, of which 27 enter into the six equations of motion of the solid.

Equations of Motion of a light thin framework of rigid wires.

12. To take the simplest possible case, let us suppose the solid to consist of a network of infinitely thin rigid massless wires through the meshes of which the liquid is circulating. If the motion of the liquid were acyclic, the wires would simply cut through the liquid without setting it in motion: hence the kinetic energy $\mathfrak{T}' + \mathfrak{T}_1$ of the acyclic motion vanishes, and the modified function becomes

$$\mathrm{H}=\xi u+\eta v+\zeta w+\lambda p+\mu q+\nu r-\mathrm{K}, \quad . \quad . \quad . \quad . \quad (32)$$

a result otherwise evident from the fact that $\mathfrak{T}_1$ only involves surface integrals taken over the infinitely small surface of the solid, while $\xi, \eta, \zeta, \lambda, \mu, \nu$ being integrals taken over the finite surfaces of barriers are in general finite.

If we choose as our axis of x the Poinsot's central axis of the impulse whose six components are $\xi, \eta, \zeta, \lambda, \mu, \nu$, the modified function will reduce to the form

$$\mathrm{H}=\Xi u+\Lambda p-\mathrm{K}. \quad . \quad . \quad . \quad . \quad . \quad (33)$$

If there is only one aperture, $\xi, \eta, \zeta, \lambda, \mu, \nu$ are all proportional to the circulation κ and the central axis of the impulse is fixed in position relative to the solid: if there are several apertures the position of the axis depends on the ratios of the circulations through the various apertures, but throughout the motion it in every case remains fixed relatively to the solid.

The six equations of motion (19) (20) now reduce to

$$\left.\begin{array}{ll} X=0, & L=0, \\ Y=r\Xi, & M=w\Xi+r\Lambda, \\ Z=-q\Xi, & N=-v\Xi-q\Lambda. \end{array}\right\} \quad . \quad . \quad . \quad (34)$$

Since these equations do not involve u or p, we see that *no forces will have to act on the solid in order to maintain a screw motion whose axis coincides with the central axis of the impulse.*

13. To interpret the equations still further, let us suppose that u and p are both zero, since they do not enter into the equations of motion. Then the motion whose components are $(0, v, w, 0, q, r)$ consists of two screws whose axes are the axes of y and z respectively, and, by the theory of screws, these are equivalent to a single screw whose axis is a certain straight line intersecting the axis of x and perpendicular to it. We may take this straight line as our axis of z, for hitherto we have only fixed the position of the axis of x. We have then

$$v=0, \quad q=0.$$

The equations (34) therefore reduce to

$$\left.\begin{array}{ll} X=0, & L=0, \\ Y=r\Xi, & M=w\Xi+r\Lambda, \\ Z=0, & N=0. \end{array}\right\} \quad . \quad . \quad . \quad (35)$$

Hence the solid is acted on by a wrench (Y, M) whose axis is the axis of y. Thus the axis of the impressed wrench is perpendicular to the central axis of the impulse of the fluid motion, and to the axis of the screw motion of the body.

Let Π be the pitch of the impulse, ϖ the pitch of the screw motion of the solid, P the pitch of the impressed wrench, then

$$\Pi=\frac{\Lambda}{\Xi}, \quad \varpi=\frac{w}{r}, \quad P=\frac{M}{Y},$$

and therefore by (35),

$$P=\varpi+\Pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

is the relation connecting the three pitches.

In particular, if $r = 0$ the equations of motion give

$$Z=0, \quad M=w\Xi,$$

showing that a couple M about the axis of y will produce translational motion with velocity M/Ξ along the axis of z.

14. More generally, let the motion be a screw motion about an axis whose inclination to the axis of x is θ and whose shortest distance from that axis is a. Take this shortest distance as the axis of y, and let the screw motion consist of a linear velocity V combined with an angular velocity Ω, the pitch V/Ω being denoted, as before, by ϖ.

It will be readily found that the six components of the screw motion are

$$\left.\begin{array}{ll} u=V\cos\theta+\Omega a\sin\theta, & p=\Omega\cos\theta, \\ v=0, & q=0, \\ w=V\sin\theta-\Omega a\cos\theta, & r=\Omega\sin\theta, \end{array}\right\}; \quad . \quad . \ (37)$$

so that the equations (34) now give

$$\left.\begin{array}{ll} X=0, & L=0, \\ Y=\Omega\Xi\sin\theta, & M=V\Xi\sin\theta-\Omega a\Xi\cos\theta+\Omega\Lambda\sin\theta, \\ Z=0, & N=0, \end{array}\right\}.(38)$$

The impressed wrench therefore has for its axis the shortest distance between the axis of the screw motion of the solid and the axis of the impulse of the cyclic fluid motion. To find the pitch of the wrench, we have, by division,

$$\frac{M}{Y}=\frac{V}{\Omega}-a\cot\theta+\frac{\Lambda}{\Xi},$$

that is,

$$P=\varpi-a\cot\theta+\Pi. \quad . \quad . \quad . \quad . \quad . \ (39)$$

15. In the case of a fine massless circular ring Λ vanishes, or the impulse of the cyclic motion is purely translational. For it is clear that the axis of the ring is the axis of this impulse (the above axis of x), also the fluid motion will evidently be unaffected by rotating the ring about its axis; and therefore the modified function is independent of the angular velocity p.

The equations (34) now become

$$\left.\begin{array}{ll} X=0, & L=0, \\ Y=r\Xi, & M=w\Xi, \\ Z=-q\Xi, & N=-v\Xi. \end{array}\right\} \quad . \quad . \quad . \ (40)$$

Hence a constant force Y along the axis of y causes uniform rotation with angular velocity Y/Ξ about the axis of z, and a constant couple M about the axis of y causes uniform translational velocity M/Ξ along the axis of z.

It is to be noticed that the impressed wrench never does work in the resulting screw motion, in accordance with the principle of Conservation of Energy.

16. The above results show the effective forces produced by circulation of the fluid on any perforated solid whatever. In the general case the modified function contains the quadratic terms $\mathfrak{T}'+\mathfrak{T}_1$ in addition to the terms of the first degree considered in the above investigation. If we suppose that the solid is moving in any given manner, the six equations of motion (19, 20) determine the components of the impressed wrench (X, Y, Z, L, M, N) necessary to maintain the given motion. This impressed wrench may be divided into two parts, one being due to the terms $\mathfrak{T}'+\mathfrak{T}_1$ in the modified function, the other being due to the terms

$$\xi u+\eta v+\zeta w+\lambda p+\mu q+\nu r.$$

The first portion is the same as if the motion were acyclic, and represents, therefore, the wrench which would have to be impressed on the solid in order to maintain the given motion if there were no circulation. The second part represents the additional wrench which must be applied on account of the circulations, and the equations to determine it are of the forms found above.

We notice, in particular, that if the solid has any screw motion whose axis coincides with the axis of the impulse of the cyclic fluid motion, the latter wrench vanishes; so that the forces required to maintain the motion are unaffected by the circulations. In other cases the additional wrench is about an axis perpendicular to the axis of the impulse. This is true whatever be the form of the solid and the number of the circulations; but, as has already been pointed out, the position of the axis of the impulse relative to the solid is not in general independent of the circulations unless the solid has but a single aperture.

It is probable that these results might be made to furnish mechanical illustrations of certain physical phenomena; but with these we are not concerned in the present paper.

Note on the foregoing Paper.

Concerning the proper measurement of the impulse of the cyclic motion, a difficulty arises; for, as Mr. Bryan remarks, this motion cannot be set up from rest by impulses applied to the solid alone. Suppose, however, that we close each perforation by a barrier in the usual way, and let the barriers be acted on by the impulsive pressures $\kappa_1\rho$, $\kappa_2\rho$, ... respectively. And instead of these impulsive pressures being due to external forces, *suppose that they are due to some immaterial mechanism attached to the solid.* In general, an impulsive wrench must act on the solid to keep it at rest, and *this wrench is the required impulse of the cyclic motion*; for the only other impulses acting on the system are due to the mutual reactions of the solid and fluid, exerted partly over the surface of the solid and partly through the barriers and attached mechanism, and such mutual reactions cannot affect the impulse. The wrench thus found is of course the same as would be obtained by supposing the impulses on the barriers to be due to *external* impulsive forces, and compounding with these the impulse *then* necessary to hold the solid at rest. This is in agreement with Prof. Lamb's investigation, which Mr. Bryan has quoted.

More generally, if the solid is in motion, and the liquid is also circulating, we may suppose the instantaneous motion to have been set up from rest by an immaterial mechanism connecting the barriers with the solid at the same time that the requisite external impulses act on the solid. The resultant of these last is, as before, the impulse of the whole motion, and is identical with that found by supposing the barriers actuated by impulses *from without*, and compounding with these the impulse *then* necessary to give to the solid its instantaneous motion.

The same point may be further illustrated by supposing the circulations κ to vary continuously during the motion. To effect this variation we may suppose finite uniform pressures, $P_1 \ldots P_m$, to be exerted over certain ideal surfaces which occupy the positions of barriers. The rate of variation $\dot{\kappa}$ of any circulation is given by $P=\dot{\kappa}\rho$, and in order that it may take place without the direct operation of external forces and couples we may conceive the pressure P to be due, as before, to some highly idealized mechanism attached to the solid. As before, the only forces capable of modifying the impulse are the external forces acting on the solid; and the equations of motion are therefore still to be found by equating the impressed force- and couple-components to the corresponding variations of the "impulse." Since we know the expressions for the impulse-components corresponding to a given instantaneous motion of the solid and given circulations, we have only to remember that in these expressions the κ's are functions of the time, and, just as before, the equations of motion are directly deducible from Hayward's formulæ. Equations (19) (20) of Mr. Bryan's paper will thus be applicable to the present case, provided that in the value of H given by (27) the κ's are allowed to vary.

An investigation proceeding from a consideration of the impulse of the whole motion is not so entirely satisfactory, I think, as the direct method given by Mr. Bryan; but, at the same time, this brief attempt to interpret the impulse of the cyclic motion may not be without interest.—C. V. BURTON.

Note added by the Author.

Dr. Burton's note is of much value as showing more exactly what is meant by the "impulse" of the motion in the ordinary investigations given by Prof. Lamb, and, in a less intelligible form, by Basset.

The equations of motion under finite forces may be deduced by equating the change of momentum in a small time-interval δt to the impulse of the impressed forces, taking into account the fact that in the interval δt the origin has a displacement of translation $(u\delta t, v\delta t, w\delta t)$ and the axes have rotational displacements $(p\delta t, q\delta t, r\delta t)$, so that the final momenta are referred to a different set of axes to the original momenta.

The mode of forming the equations of motion is given by Prof. Greenhill (*Encyclopædia Britannica*, art. "Hydromechanics") for the case of acyclic motion, but it is hardly so obvious *why* in thus forming the equations of motion of a perforated solid, it is necessary to include in the "impulse" terms representing the components of the wrench applied to the *barriers* as well as to the solid. We may, however, suppose the changes which actually occur in the time δt to have been produced as follows :—

1st. Let the solid and fluid be reduced to rest by an impulsive wrench applied to the solid, and transmitted to a series of barriers crossing the perforations. The components of this wrench will be found to be

$$\frac{\partial T}{\partial u}+\xi, \text{ \&c.}, \ldots \frac{\partial T}{\partial p}+\lambda, \text{ \&c.} \ldots$$

2nd. The barriers being rigidly connected with the solid, let the latter receive small displacements whose translational and rotational components are ($u\delta t$, $v\delta t$, $w\delta t$, $p\delta t$, $q\delta t$, $r\delta t$) and let the solid come to rest in its new position. The fluid will evidently also come to rest, and therefore no impulse will be impressed on the system by this change (as may be otherwise seen by supposing the change to take place very slowly).

3rd. Let the solid be set in motion with velocity-components ($u+\partial u/\partial t\,.\,\delta t, \ldots \quad p+\partial p/\partial t\,.\,\delta t \ldots$) referred to the new positions of the axes, and let the circulations κ be started in the new position of the solid by a suitable impulsive wrench applied to the solid and transmitted from it to the barriers.

Then the impulse of the impressed forces (components $X\delta t \ldots$, $L\delta t \ldots$) is the resultant of the wrenches required to stop the whole system in the first process and to start it again in the third.

It is, therefore, that impulse which must be compounded with the total impulse in the initial position in order to obtain the total impulse in the final position.

Whence Hayward's equations of motion follow at once (as shown in Greenhill's article above referred to), and they take the form of the above equations (14), (15).

If we were merely to stop the solid in the first process without stopping the liquid, the cyclic motion would cause the liquid to exert a pressure on the solid in the second process, and the impulse of this pressure would not be zero, but would have to be taken into account in forming the equations of motion. It would be wrong, therefore, to deduce the equations of motion from the impulse applied to the solid alone, as is evident in the analogous case of a solid containing one or more gyrostats.

Discussion.

The President said the author had done good service by attacking the difficult problem by elementary methods. He had also arrived at some very interesting conclusions, particularly the one showing that a perforated body moving through a liquid required no force to keep up the motion.

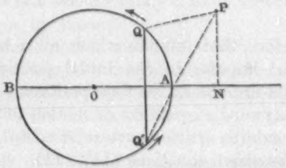
XIII. *The Magnetic Field of a Circular Current.*

By Professor G. M. Minchin, *M.A.*[*]

Clerk Maxwell gives a method of drawing the lines of magnetic force due to a circular current ('Electricity and Magnetism,' Art. 702) by means of a series of circles and a series of parallel lines. The object of the following paper is to show how these curves can be described by a slightly different method, and to exhibit the geometrical connexion of the series of circles.

Let AQBQ' be the circular current whose sense is indicated by the arrows, the plane of the circle being that of the paper;

Fig. 1.



let P be any point in space, PN the perpendicular from P on the plane of the circle, and NAOB the diameter of the circle drawn through N. We shall calculate the *vector potential* of the current at P.

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[*] Read March 10, 1893.

Draw any ordinate, QQ′, of the circle perpendicular to BA; and consider two equal elements of length of the circle, each equal to ds, at Q and Q′. Resolving each of these along and perpendicular to QQ′, we see that the latter components are in opposite senses, and hence their vector potentials at P cancel each other, since $PQ = PQ'$. If ψ is the angle QOA, $a =$ radius of circle, $i =$ current strength, the components of ids along Q′Q being equal and in the same sense, the two elements of current at Q′ and Q conspire in giving a vector potential $\frac{2i \cos\psi}{PQ}.ds$ perpendicular to the plane PON. Hence the total vector potential at P is perpendicular to the plane PON. If, therefore, OA is the axis of x, the perpendicular at O to the plane of the circle the axis of z, and the diameter at O perpendicular to AB the axis of y, the components of the vector potential being, as usual, denoted by F, G, H, the only component existing is G; but, by taking the components of the vector potential at a point indefinitely close to P in the direction of the axis of y, we easily find that

$$\frac{dF}{dy} = -\frac{G}{\alpha}.$$

Hence if X, Y, Z are the components of the force of the current per unit magnetic pole at P, since this force is the curl of the vector potential, we have

$$X = -\frac{dG}{d\gamma}, \quad Y = 0, \quad Z = \frac{dG}{d\alpha} + \frac{G}{\alpha},$$

where α (=ON) and γ (=NP) are the coordinates of P. If along the line of force at P the increments of the coordinates are $\Delta\alpha$, $\Delta\gamma$, we have

$$\frac{\Delta\alpha}{\Delta\gamma} = \frac{X}{Z}.$$

Hence along this line we have

$$\frac{dG}{d\alpha}\Delta\alpha + \frac{dG}{d\gamma}\Delta\gamma + \frac{G}{\alpha}\Delta\alpha = 0,$$

i. e., $G.\alpha =$ constant along the line of force.

We shall therefore calculate the vector potential, G, at P.

Evidently

$$G = 2i\int_0^\pi \frac{\cos\psi}{PQ} \,.\, ds$$

$$= 2ia\int_0^\pi \frac{\cos\psi \,.\, d\psi}{\sqrt{\alpha^2+a^2+\gamma^2-2a\alpha\cos\psi}}$$

$$= \frac{i}{\alpha}\int_0^\pi \left(\frac{\alpha^2+a^2+\gamma^2}{D}-D\right)d\psi,$$

denoting the denominator by D. Now let $\psi=\pi-\omega$, and let $\rho^2=(\alpha+a)^2+\gamma^2$, $\rho'^2=(\alpha-a)^2+\gamma^2$, so that $\rho=PB$, $\rho'=PA$. Then

$$G = \frac{i}{2\alpha}\int_0^\pi \left\{\frac{\rho^2+\rho'^2}{\rho} \,.\, \frac{1}{\Delta}-2\rho\Delta\right\}d\omega,$$

where

$$\Delta = \sqrt{1-\left(1-\frac{\rho'^2}{\rho^2}\right)\sin^2\frac{\omega}{2}}.$$

Let $\omega=2\phi$, and $k^2=1-\frac{\rho'^2}{\rho^2}$; then, finally,

$$G = \frac{4ia}{\rho}\left\{\frac{2}{k^2}(K-E)-K\right\},$$

where K and E are the complete elliptic integrals of the first and second kinds with modulus k; so that the quantity in brackets is a function of the ratio $\frac{PA}{PB}$ simply.

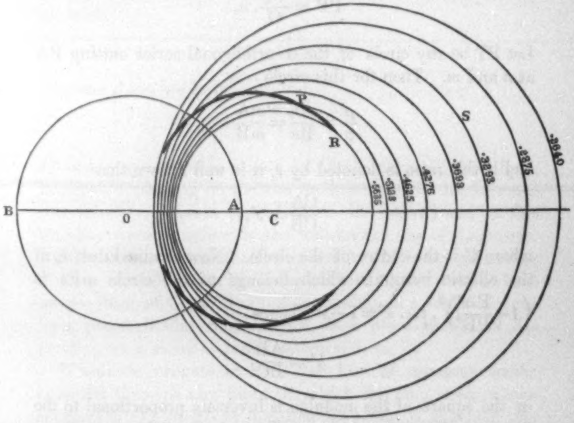
Also, since $\rho^2-\rho'^2=4a\alpha$, we have $\alpha=\frac{\rho^2k^2}{4a}$, and the quantity $G \,.\, \alpha$ which is constant along the line of force is given by the equation

$$G \,.\, \alpha = i\rho\{2(K-E)-k^2K\}.$$

It is thus seen that at every point in space G is of the form $\frac{1}{\rho}f\left(\frac{\rho'}{\rho}\right)$; so that at all points on the surface for which $\frac{\rho'}{\rho}$ is a constant, the value of G will vary inversely as ρ. The surface for which $\frac{\rho'}{\rho}$ is constant is a sphere having its centre on the line BA produced and cutting the sphere having BA for diameter orthogonally. If we assign a series of values to the ratio $\frac{\rho'}{\rho}$, we obtain a series of spheres having their centres on BA and cutting the given sphere orthogonally, the radius of each sphere of the series being, therefore, the length of a tangent from its centre to the sphere described on BA; for, given the base, BA, of a triangle, and the ratio of the sides, the locus of the vertex is a circle whose diameter is the join of the points which divide BA internally and externally in the given ratio. The surface locus of the vertex is the sphere generated by the revolution of this circle.

On account of the symmetry of the current round its axis through O, the lines of force and those of constant vector potential are the same in all planes through the axis.

Fig. 2.



We may, then, confine our attention to the plane PON, and suppose fig. 2 to be in this plane, the current being in this figure represented in projection by the line BA. Describe a series of circles having their centres on BA produced and cutting the circle described on BA as diameter orthogonally. Along each of these circles, then, the ratio $\frac{\text{PA}}{\text{PB}}$ is constant, P being any point on the circle.

Consider first the lines of constant vector potential. For each of the circles let the value of the quantity $\frac{2}{k^2}(\text{K}-\text{E})-\text{K}$ be calculated. Denote this quantity by Q for any one circle; then

$$\text{G}=\frac{4\text{Q}ia}{\rho};$$

so that if we wish to trace out the line of constant vector potential for which G has any given value, we can find the point, P, in which it cuts any circle of the series by measuring the length PB such that

$$\text{PB}=\frac{4\text{Q}i}{\text{G}}.\,a.$$

Let PT be any circle of the co-orthogonal series cutting BA at n and m. Then for this circle

$$\frac{\rho'}{\rho}=\frac{\text{A}n}{\text{B}n}=\frac{m\text{A}}{m\text{B}};$$

and if this ratio is denoted by s, it is well known that

$$\frac{\text{CA}}{\text{CB}}=s^2,$$

where C is the centre of the circle. Now the modulus, k, of the elliptic integrals which belongs to the circle $m\text{P}n$ is $\left(1-\frac{\text{PA}^2}{\text{PB}^2}\right)^{\frac{1}{2}}$, *i.e.* $k^2=1-s^2$; hence

$$k^2=\frac{\text{AB}}{\text{BC}},$$

or the square of the modulus is inversely proportional to the distance, BC, of the centre of the circle from B.

The circles employed by Clerk Maxwell in drawing the lines of force can be easily shown to be this co-orthogonal system whose centres are ranged along BA produced. For, his rule is to assign a series of values to θ, and construct a series of circles whose centres lie on BA, the radius of each being $\frac{a}{2}(\operatorname{cosec}\theta - \sin\theta)$, while the distance of its centre from O is $\frac{a}{2}(\operatorname{cosec}\theta + \sin\theta)$; the modulus belonging to this circle is $\sin\theta$. For the series of circles he then calculates the values of the expression (constant for each circle) $\frac{\sin\theta}{(K-E)^2}$, and the point on each circle which lies on any assigned line of force is found by drawing a certain right line perpendicular to BA. It is at once found that this series of circles is precisely the co-orthogonal system above described; but Clerk Maxwell's modulus is not the same function of the ratio $\frac{\rho'}{\rho}$, or of the radius of the circle selected, as that adopted above; for, with Clerk Maxwell, if r is the radius of any circle of the series and k the corresponding modulus,

$$r = \frac{a}{2}\left(\frac{1}{k} - k\right),$$

whereas above we have

$$r = \frac{2\sqrt{1-k^2}}{k^2} \cdot a.$$

Of course (as stated in a note by Clerk Maxwell) the elliptic integrals depending on the one modulus can be transformed into elliptic integrals depending on the other; and in this case the transformation is the well-known one of Lagrange. But the constructions for the points in which any given line of force cuts the series of circles will not be the same in both cases—those of Clerk Maxwell depending on a series of right lines perpendicular to BA, and those above indicated depending on a series of radial distances from B.

When we propose to draw the line of constant vector potential through any point, P, which lies on a circle whose constant is Q_0, let PB be ρ_0; then the point, R, in which this line meets any other circle, whose constant is Q, is found from the relation

$$\rho = \rho_0 \frac{Q}{Q_0},$$

where $\rho = BR$.

This latter method has a certain advantage for the eye, inasmuch as it enables us to see readily those circles of the series outside which the line of constant vector potential through any proposed point lies.

Consider now the lines of force. With the above value of Q, the quantity which is constant along a line of force is $\rho \, . \, k^2 Q$, so that on each of the above circles in fig. 2 we must now mark the number k^2Q. Denote this by Q'. Then the above relation for points on the same line of constant vector potential becomes for the lines of force

$$\rho = \rho_0 \frac{Q_0'}{Q'},$$

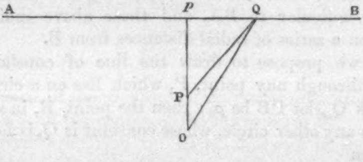
and the construction proceeds in the same way. The constants, Q', for the above series of circles, beginning at the innermost, are :—

·4841 ; ·4301 ; ·3775 ; ·3396 ; ·2782 ; ·2376 ; ·1954 ; ·1727.

The values of the Q's diminish outwards for the circles; so that if we consider the line of vector potential at any point, S, suppose, which is such that SB is greater than the distance from B of the point along AC in which any circle interior to that passing through S cuts the line BAC, it is at once obvious that the line of vector potential which belongs to S is wholly outside all such circles. The numerical values of Q for the circles in fig. 2 are marked at the circumferences, and as much of the line of potential belonging to P is drawn as is justified by the number of circles represented in the figure.

The fundamental proposition of electromagnetism is that the intensity of magnetic force produced at any point in presence of electric currents is the curl of the vector potential at the point. But if in the field there is a current in an

Fig. 3.



infinitely long straight wire, AB, we find that at every point in the field the vector potential due to this current is infinite. Hence it seems impossible to deduce the magnetic force, and the lines of magnetic force, from the above fundamental proposition. This result is unsatisfactory, and it manifestly points to some defect in our definition of the vector potential.

We are presented with a similar unsatisfactory result in the general theory of gravitation potential. Thus, taking the common definition of gravitation potential, if AB is a limited uniform bar attracting according to the law of inverse square, we know that the potential which it produces at any point, P, is proportional to $\log\left(\cot\frac{A}{2}\cot\frac{B}{2}\right)$, where $A=\angle PAB$, $B=\angle PBA$. Now, if the rod extends to infinity, this expression becomes infinite. I have shown ('Statics,' vol. ii. Art. 332) how this difficulty arises, and how it is to be remedied by mending the definition of potential. The difficulty is avoided in a similar manner with regard to the vector potential.

Thus, since we are concerned only with differential coefficients of the vector potential, the ordinary components, F, G, H, of this vector may have added to them any *constant* quantities whatever. This amounts to saying that the vector potential at any variable point, P, in the field is the vector potential at any *fixed* point, O, plus the vector difference between P and O. It does not matter whether the vector at O is infinite or not: it is a constant in the field. As in the general gravitation field we are concerned with *differences* of potential only, so in the electromagnetic field we are concerned with vector differences only.

Let us, then, calculate for the infinite straight current AB the vector difference between P and a point O on the perpendicular, Pp, at a constant distance $Op=a$ from the line.

Let $Pp=r$, and let an element, ds, of the line AB be taken at any point, Q; let $\angle pPQ=\theta$. Then the vector difference, due to this element, at P is $\frac{ds}{QP}-\frac{ds}{QO}$, or

$$\left\{1-\frac{r}{\sqrt{r^2+(a^2-r^2)\cos^2\theta}}\right\}\cdot\frac{d\theta}{\cos\theta}.$$

Double the integral of this from $\theta=0$ to $\theta=\frac{\pi}{2}$ is the vector difference at P due to unit current in AB. Expanding the radical in ascending powers of $\lambda\left(=\frac{a^2-r^2}{r^2}\right)$, we have the vector equal to

$$2\int_0^{\frac{\pi}{2}}\left\{\tfrac{1}{2}\lambda-\frac{1.3}{2.4}\lambda^2\cos^2\theta+\frac{1.3.5}{2.4.6}\lambda^3\cos^4\theta-\ \ldots\right\}d(\sin\theta)$$

$$=\lambda-\tfrac{1}{2}\lambda^2+\tfrac{1}{3}\lambda^3-\tfrac{1}{4}\lambda^4+\ \ldots$$

and this $=\log_e(1+\lambda)=2\log\frac{a}{r}$. Thus, then, the vector difference at any point, P, is measured by

$$C-2\log r,$$

where C is a constant; and this gives the known value of the magnetic force at P, viz., $-\frac{dG}{dr}$ (where G is the vector potential), perpendicular to the plane PAB, *i.e.* $\frac{k}{r}$, where k is a constant. In this way, then, the inconvenience of dealing with an infinite vector potential in presence of an infinitely long (or very long) straight current is avoided.

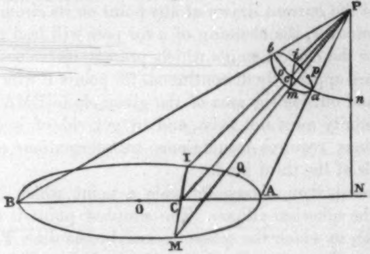
The lines of constant magnetic potential, or the loci of points, P, at which the given circular current subtends a constant conical ("solid") angle, are the orthogonal trajectories of the lines of force, and can be drawn when these lines are drawn.

It is not easy to draw these equipotential curves independently, or even to deduce their typical equation from that of the lines of force by the mathematics of orthogonal trajectories.

The magnitude of the conical angle subtended at any point by a given circle can be expressed in finite terms by means of complete elliptic integrals of the third kind. The parameter involved in these integrals will depend on the way in which they are taken.

If a sphere of unit radius is described round P as centre, and lines are drawn from P to the points on the circumference of the given circle, BMAI, fig. 4, these lines will intercept on the sphere a spherical ellipse, *bmai*, whose area is the conical angle subtended by the circle at P. The minor axis of this ellipse is the great circular arc *ab* determined by the lines PA, PB, while the major axis, *mi*, is determined by the chord, MI, of the circle which subtends a maximum angle, MPI, at P. This line is determined by drawing the bisector,

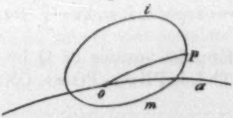
Fig. 4.



PC, of the angle BPA, meeting BA in C; then MI is the chord through C perpendicular to the plane BAP. The point *c* in which PC meets the surface of the sphere is the centre of the spherical ellipse.

Now, given any curve, *mpi*, fig. 5, on a sphere of unit

Fig. 5.



radius, its area is $\int(1-\cos\theta)d\phi$, where, if *o* is any point on the sphere *inside* the area, θ is the circular measure of the spherical radius vector *op* drawn to any point, *p*, of the curve, and ϕ is the angle between the radius *op* and any fixed arc, *oa*, drawn at *o*. If, as said, the pole *o* is inside the area, ϕ goes from *o* to 2π; but if *o* is *outside* the curve, the area has a different expression, viz.:

$$\int_0^o \cos\theta d\phi,$$

the longitude angle ϕ obviously starting and ending with a zero value. If o is *on* the curve, the expression for the area is again different.

In calculating the area of the above ellipse it would be natural to choose for pole (o) the point n in which the sphere is cut by the line PN; but this leads to difficulties when the position of P is such that n falls on the ellipse. This will happen when P is on any perpendicular to the plane of the circle of the current drawn at any point on its circumference; and, moreover, the choosing of n for pole will lead to expressions for the conical angle which present its values in forms which are apparently discontinuous for points P which project inside and outside the area of the given circle BMAI. Such discontinuity must not exist, and to get rid of it from the expressions requires troublesome transformations of elliptic integrals of the third kind.

We must, then, choose for pole a point which is always inside the spherical ellipse. The simplest point is the point o (fig. 4), in which the sphere is cut by the line PO which joins P to the centre, O, of the given circle. This point is, of course, always inside the ellipse.

Let, then, Q be any point on the given circuit, and p the point in which PQ cuts the ellipse. Taking for the fixed plane of longitude through o the plane baP, or BAP, and denoting the angle poa by ϕ, the area of the ellipse is

$$\int_0^{2\pi}(1-\cos op)d\phi, \text{ i. e., } 2\pi-\int_0^{2\pi}\cos op\,.\,d\phi.$$

Denoting, as before, the position of Q by the angle ψ, or QOA, we easily find, if PN$=z$, PO$=r$, ON$=x$,

$$d\phi=\frac{rz}{z^2+x^2\sin^2\psi}\,.\,d\psi,$$

$$\cos op=\frac{r^2-ax\cos\psi}{r\sqrt{r^2+a^2-2ax\cos\psi}},$$

Hence

$$\int_0^{2\pi}\cos op\,.\,d\phi=2z\int_0^{\pi}\frac{r^2-ax\cos\psi}{\sqrt{r^2+a^2-2ax\cos\psi}}\,.\,\frac{d\psi}{z^2+x^2\sin^2\psi}.$$

Putting $\psi=\pi-\chi$, this becomes

$$2z\int_0^{\pi}\left\{\rho\sqrt{1-k^2\sin^2\frac{\chi}{2}}+\frac{r^2-a^2}{\rho\sqrt{1-k^2\sin^2\frac{\chi}{2}}}\right\}\frac{d\chi}{z^2+x^2\sin^2\chi},$$

where, as before, $\rho=\mathrm{PB}$, $\rho'=\mathrm{PA}$, and $k^2=1-\frac{\rho'^2}{\rho^2}$. If we put $\chi=2\omega$, this becomes

$$z\int_0^{\frac{\pi}{2}}\left(\rho\Delta+\frac{r^2-a^2}{\rho\Delta}\right)\frac{d\omega}{z^2+x^2\sin^2 2\omega},$$

where $\Delta=\sqrt{1-k^2\sin^2\omega}$.

To reduce this to elliptic integrals, we must resolve the fraction $1/z^2+x^2\sin^2 2\omega$ into two fractions. It is easily found that

$$z^2+x^2\sin^2 2\omega=z^2+4x^2(\sin^2\omega-\sin^4\omega)$$

$$=(\sqrt{z^2+x^2}+x-2x\sin^2\omega)(\sqrt{z^2+x^2}-x+2x\sin^2\omega).$$

Let ν denote the sine of the angle between PO and the axis of the current (or PN); then the expression, after resolution into partial fractions, becomes

$$\frac{z}{2r^2}\int_0^{\frac{\pi}{2}}\left(\rho\Delta+\frac{r^2-a^2}{\rho\Delta}\right)\left(\frac{1}{1-\nu+2\nu\sin^2\omega}+\frac{1}{1+\nu-2\nu\sin^2\omega}\right)d\omega.$$

The portion of this expression which has Δ in the denominator is at once the sum of two complete elliptic integrals of the third kind; and the portion which has Δ in the numerator is easily reduced to the same form. The result is

$$\frac{z}{r\rho}\int_0^{\frac{\pi}{2}}\left(\frac{r+a}{\mathrm{N}\Delta}+\frac{r-a}{\mathrm{N}'\Delta}\right)d\omega,$$

where $\mathrm{N}\equiv 1-\nu+2\nu\sin^2\omega$, and N' is the value of N when ν is changed to $-\nu$. Hence we have two elliptic integrals of the third kind, one with $\left(\frac{2\nu}{1-\nu},\ k\right)$ for parameter and modulus, and the other with $\left(\frac{-2\nu}{1+\nu},\ k\right)$. In the usual notation, then, we have for the complete expression of the conical angle subtended by the circuit at P the value

$$2\pi - \frac{2z}{\rho r}\left\{\frac{r+a}{1-\nu}\Pi\left(\frac{2\nu}{1-\nu}, k\right) + \frac{r-a}{1+\nu}\Pi\left(\frac{-2\nu}{1+\nu}, k\right)\right\}.$$

Of course it is not pretended that this expression is the most convenient for the purposes of calculation: the approximate value of the conical angle which is given by a series of spherical harmonics is that which should be employed; but it may be well to give the *complete* expression in the above form, which I have not seen published anywhere.

DISCUSSION.

Prof. Perry thought the problem had been very prettily worked out, and hoped Prof. Minchin would be able to extend the solution to cylindrical coils,—a subject in which he (Prof. Perry) had long been interested.

Mr. Blakesley said Mr. Niven had pointed out to him that the locus of points for which the ratio ρ'/ρ is constant was an anchor ring.

Prof. Minchin maintained that the locus was a sphere, and on this subject a short discussion arose, in which the President, Prof. Perry, Prof. Minchin, Dr. Burton, and Prof. S. P. Thompson took part.

Dr. Sumpner described a method of drawing the lines of force of a circular current, or of any circuit, symmetrical about an axis, by a purely experimental and graphical process. The strengths of field were first determined at several points in the plane of the current, and from these results the points through which lines of force were to pass in order that their distribution might indicate the strength of the field at all points were graphically deduced.

XIV. *On the Differential Equation of Electrical Flow.*

By T. H. BLAKESLEY, *M.A.*[*]

THE object of this paper is to point out that the theory of electrical discharge, as exemplified in the mathematical expressions employed to represent the physical facts, is incompetent to explain all the phenomena observed in extreme cases; and to show that the admission of certain properties of matter not usually recognized is the only way of satisfactorily obviating the imperfection of the existing views.

In some of the investigations I shall not employ exclusively algebraical symbolic methods, but, where it may more advantageously be adopted, I shall avail myself of the geometrical method. Such cases most frequently arise where magnitudes under consideration are capable of having negative values. All tidal effects, using the word in its most general sense, involve such magnitudes.

Electrical currents in a given conductor may have all possible values in one direction or in the opposite direction, but are otherwise restricted.

The projection of the line joining two points in space upon a fixed straight line is a geometrical magnitude of this sort. With respect to the direction in space, sometimes one of the projected points will be on one side of the second projected point, sometimes on the other. So that such a line has all the properties necessary for representing another magnitude of the same character.

In this way I shall most generally make the projection represent Electromotive Force, but occasionally Field of Magnetism at a point. As to matter of nomenclature, the only scientific term which I shall employ admitting of any doubtful interpretation, is the Effective Electromotive Force. By this term I intend to convey the idea of that electromotive force which is numerically equal to the product of the current and the resistance, at a point of time. As a department of State has recently employed the term in a totally different sense, this statement has appeared to me to be necessary in the interests of proper explanation. The effective electromotive force is the algebraical sum of all the impressed and induced electromotive forces, and is here represented by E. If V is the sum of all the impressed electromotive forces and F is the sum of all induced electromotive forces, then the equation among these quantities is $V+F=E$ universally.

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[*] Read March 24, 1893.

Geometrically, if A B, B C are lines whose projections on some one fixed straight line represent the sum of the impressed and the sum of the induced electromotive forces respectively, then the projection of A C will represent the effective electromotive force.

The three lines must form the sides of a triangle, those corresponding to the impressed and induced electromotive forces being taken the same way round the triangle, that corresponding to E being taken in the opposite direction.

Now if the actual changes in the magnitudes are harmonic, and of the same period, it is clear that the lines A B, B C, A C must remain of constant length and the triangle must rotate in its own plane at a uniform rate of such a value as to perform a complete revolution in the period of the harmonic change. The triangle thus shows admirably the way in which these magnitudes succeed one another in phase. It also follows from the properties of harmonic motion that if two magnitudes have the same harmonic period, but differ in phases by a quarter of the whole period, the corresponding lines to be projected are at right angles to each other. And hence the rate of variation of an harmonic magnitude differs in phase from the magnitude itself by a quarter of the period. But in the simplest case of a circuit being plied with an harmonic electromotive force V, it is generally considered that the induced electromotive force varies as the rate of change of the current; that is

$$F=-L\frac{dC}{dt}=-\frac{L}{R}\frac{dE}{dt}, \text{ for } E=RC,$$

where C is the current, R the resistance, and L is the coefficient of self-induction.

The equation already given then becomes

$$V - L\frac{dC}{dt} = E = RC.$$

Multiplying through by C and integrating through a complete period,

$$\int VC\,dt - L\int C\frac{dC}{dt}\,dt = R\int C^2 dt.$$

The first term represents the work done by the source of the disturbance.

The second term vanishes.

The third term represents work done in heating the circuit.

Hence the whole work done has gone to heat the circuit.

Now it is admitted on all hands that when the period is sufficiently short a radiation of energy into space takes place. A portion of this radiated energy is sometimes caught by means of a neighbouring circuit and converted into heat.

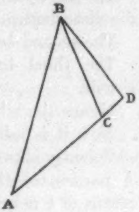
A coefficient of mutual induction and a corresponding extra term is then introduced into the equation. But are we to suppose that radiation would not proceed into space were there no neighbouring conductor? It is against probability, against the electromagnetic theory of light.

If electromagnetic waves are capable of being sent into space, we can no longer look upon the operation of establishing a current in a circuit as analogous to bending a stiff spring or displacing rigid wheelwork. The wheelwork must have indiarubber spokes or teeth.

The above equation takes no account of this radiation which is expended outside the wire, nor of any other work done elsewhere than in the conductor ; and this latter the equation states to be exactly equal to the energy expended in propagating the electromotive force. Even supposing a portion of the field is occupied by some material whose passage through a cycle of magnetization involves the loss of energy, in the form of heat, this, equally with wave-propagation through space occupied by perfectly elastic matter, will not be accounted for by the equation.

Now of such phenomena as radiation of energy in electromagnetic waves, or absorption of energy in the field, there is ample evidence. Therefore an equation will not meet such cases in which the induced electromotive force is taken as entirely in quadrature with the current, or when F is wholly of the form $-L\frac{dC}{dt}$.

Hence, in the geometrical representation it is clear that the induced electromotive-force line must not be exactly at right angles to that of the effective electromotive-force line; *i. e.* the angle B C A is not exactly a right angle; and it is easy to see that it must be greater than a right angle, for B C may be resolved into B D . D C, where B D C is a right angle and A C D is one straight line. For then the whole work done is equal to $\frac{AD . AC}{2R}$. The work done in heating the conductor is $\frac{AC . AC}{2R}$, and the difference, or the work done in the field, is $\frac{AC . DC}{2R}$.



Hence, if D lies on the side of C nearer to A, A D would be less than A C, and the work done by the discharge would be less than that required to heat the conductor: in other words, energy would have to be received from space.

Hence the induced electromotive forces may be represented by two components—one B D in quadrature with the current, and one D C in opposition to it,

$$-L\frac{dC}{dt}-\lambda C,$$

where λ may or may not be a constant, but is in kind a resistance.

The equation among the electromotive forces may be written

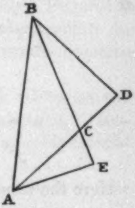
$$V-L\frac{dC}{dt}-\lambda C=RC.$$

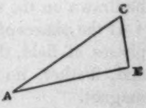
Multiplying through by $C dt$ and integrating through a complete period,

$$\int V C dt - L \int C \frac{dC}{dt} dt = R \int C^2 dt + \int \lambda C^2 dt.$$

The second term on the left vanishes as before, the first term representing the whole work done. On the right the first term heats the conductor and the second term gives energy to space.

We may go somewhat further into the causes of such an induced electromotive-force component if we employ the geometrical mode of symbolizing the electric quantities. BC, the induced electromotive-force line, should be at right angles to the induction through the circuit, for it is the rate of increase of the latter which produces the former. Hence if A E is a perpendicular let fall upon B C produced, A E will represent the phase of the magnetic induction. But A C being in phase with the current is in phase with the field. Hence E A C, or C B D which is equal to it, is a magnetic phase-lag, and A E may be said to be in phase with the *effective* field, and therefore with the induction. This suggests that if we employ the lower lines of the figure to represent *fields*, we may make up a triangle A C E such that A C is the impressed field, C E an induced field, and A E an effective field, of course when, as usual, projected on a fixed line; C E being perhaps, though by no means certainly, at right angles to A E. However, whether C E here has in any case two components perpendicular and parallel respectively to A E or not, it appears very certain that the perpendicular component must exist. Assuming at first that it alone exists,—





If we employ small letters :—

v for impressed field $= AC$,
f for induced field $= CE$,
e for effective field $= AE$,

l = coefficient of magnetic self-induction, so that

$$f=-l\frac{d\mathrm{I}}{dt};$$

and μ for the permeability, I for the rate of magnetic induction, *i. e.* per square centim., we have

$$v-l\frac{d\mathrm{I}}{dt}=e=\frac{\mathrm{I}}{\mu}.$$

To obtain an equation of energy from this we must multiply (not by I, as analogy would at first sight perhaps dictate) by $\frac{d\mathrm{I}}{dt}.dt$ × cross section, for the formula for energy is

$$[l^2t^{-2}m]\equiv[\mu^{-\frac{1}{2}}l^{-\frac{1}{2}}t^{-1}m^{\frac{1}{2}}][\mu^{\frac{1}{2}}l^{\frac{5}{2}}t^{-2}m^{\frac{1}{2}}]t$$

$$\equiv\Big[\text{Field}\Big]\Big[\frac{\text{lines}}{t}\Big]t$$

$$\int v\frac{d\mathrm{I}}{dt}.dt-l\int\Big(\frac{d\mathrm{I}}{dt}\Big)^2dt=\int\frac{\mathrm{I}}{\mu}\frac{d\mathrm{I}}{dt}.dt.$$

Here the term on the right hand disappears necessarily, and the work expended, if any, is equal to $l\int\Big(\frac{d\mathrm{I}}{dt}\Big)^2dt$. Hence this work vanishes only if $l=0$, *i. e.*, if there is no component field in quadrature with the induction; a curious antithesis to the electric problem. If there were a field induced in phase with the induction, it would not result in the dissipation of energy. No argument for such a state of things can be drawn on the score of loss of energy.

If the phases of magnetism in any cycle coincided with the phases of field, there could be no such thing as hysteresis; and, further, no radiation of energy from an alternating magnet.

But both these phenomena have been for many years recognized. It follows, therefore, that when induction through any space changes, a magnetic field is induced acting counter to the change.

Now it may be noticed that the tangents of the angles of lag, whether electric or magnetic, involve a coefficient in their numerator and the value of the period in their denominator. They therefore become larger as the period is made less. It might therefore happen that extreme rapidity of change would be necessary before a lag of current or of induction could be detected. Electric lags, or, at all events, the coefficients of self-induction can readily be measured. Magnetic lags have been measured by the author in certain cases by special artifices, but when we deal with a medium of small permeability, as air, the period must be extremely minute to make the lag-angle sensible, and as yet no machines possess a sufficient frequency to effect it. Recourse has been had to the rapid oscillations which take place when a Leyden jar is discharged. In these cases the radiation has been frequently caught and approximately measured, and it is therefore in these very cases that the rectification of the formula becomes important.

I propose to investigate by geometry and otherwise the conditions under which a Leyden jar is discharged. Geometry especially will afford an excellent and graphic insight into the question of the oscillatory discharge.

As in the case of the sustained discharge, I shall first take the usual formula, obtain the geometrical illustration of it, and, observing where the defect shows itself, pass by an easy transition to the truer state of things.

In the case of the discharge of a condenser through a simple circuit removed from proximity with any other, we have the following equations to deal with ;—

The general equation $V+F=E, \quad \text{(i.)}$

$$E = RC, \qquad \text{(ii.)}$$

$$F = -L\frac{dC}{dt}, \qquad \text{(iii.)}$$

$$C = -K\frac{dV}{dt}, \qquad \text{(iv.)}$$

where

C is the current discharging the condenser ;
K is the capacity of the condenser ;
L is the coefficient of self-induction ;
R is the resistance ;
F is the induced electromotive force ;

E is the effective electromotive force ;
V is the potential difference of the plates of the condenser.

Eliminating E and F from (i.) by means of (ii.) and (iii.),

$$V-L\frac{dC}{dt}-RC=0. \quad . \quad . \quad . \quad . \quad . \quad (v.)$$

Differentiating,

$$\frac{dV}{dt}-L\frac{d^2C}{dt^2}-R\frac{dC}{dt}=0,$$

and substituting for $\frac{dV}{dt}$ from (iv.),

$$-\frac{C}{K}-L\frac{d^2C}{dt^2}-R\frac{dC}{dt}=0,$$

or

$$C+KR\frac{dC}{dt}+LK\frac{d^2C}{dt^2}=0,$$

the differential equation of C.

Since $C=\frac{E}{R}$, it follows that the equation

$$E+KR\frac{dE}{dt}+LK\frac{d^2E}{dt^2}=0$$

is also true, and therefore is the differential equation of E.

Again, from (v.) and (iv.), since $\frac{dC}{dt}=-K\frac{d^2V}{dt^2}$, obtained by differentiating (iv.),

$$V+RK\frac{dV}{dt}+LK\frac{d^2V}{dt^2}=0.$$

This is the differential equation of V.

Thirdly, differentiating twice the equation

$$V+F-RC=0,$$

$$\frac{d^2V}{dt^2}+\frac{d^2F}{dt^2}-R\frac{d^2C}{dt^2}=0,$$

from (iii.) by differentiation it is seen that

$$\frac{d^2C}{dt^2}=-\frac{1}{L}\frac{dF}{dt},$$

and from (iv.),

$$\frac{d^2V}{dt^2} = -\frac{1}{K}\frac{dC}{dt},$$

which is further reduced to $\frac{1}{K}\frac{F}{L}$ by (iii.).

Hence $$\frac{F}{KL} + \frac{d^2F}{dt^2} + \frac{R}{L}\frac{dF}{dt} = 0,$$

or

$$F + KR\frac{dF}{dt} + KL\frac{d^2F}{dt^2} = 0,$$

the differential equation of F.

It is thus clear that the variables V, F, E, C all have the same form of differential equation, viz. :—

$$E + KR\frac{dE}{dt} + KL\frac{d^2E}{dt^2} = 0.$$

Of course, to make this equation homogeneous,

KR is of the order (time) ;
KL " " (time)^2.

KL may be written $KR \cdot \frac{L}{R}$, or still better $\frac{KR}{2} \cdot \frac{2L}{R}$.

If we write $$\frac{2L}{R} = t_1$$

and $$\frac{KR}{2} = t_3,$$

t_1 and t_3 are time-constants of the circuit, and the differential equation may be written

$$E + 2t_3\frac{dE}{dt} + t_1 t_3\frac{d^2E}{dt^2} = 0.$$

In this expression the two time-constants may be considered to be independent.

To obtain a geometrical representation of the changes :—

(1) Suppose a line, whose length is r, to shrink logarithmically so that its change is represented by the equation

$$\frac{dr}{dt} = -\frac{r}{t_1},$$

where t_1 is a time-constant.
Then

$$r = ae^{-\frac{t}{t_1}},$$

where a is the value of r at the beginning of the time, and t_1 appears as the time taken for r to shrink to $\frac{1}{e}$ of itself.

(2) Secondly, suppose a straight line in a plane to constantly change in *direction* at a uniform rate, in the same sense. If θ is the angle measured from a fixed direction,

$$\frac{d\theta}{dt} = \frac{2\pi}{t_2},$$

where t_2 is a time-constant. Hence

$$\theta = \frac{2\pi}{t_2} \cdot t.$$

Whence t_2 appears as the time required to describe 2π.

(3) Suppose a line to undergo both the changes contemplated, which is possible, since one is a change of length, the other a change in direction. Then, eliminating the time, we have

$$r = ae^{-\frac{\theta}{\frac{2\pi t_1}{t}}},$$

or $r = ae^{-\frac{\theta}{\tan\beta}}$, where $\tan\beta = \frac{2\pi t_1}{t_2}$.

This is the equation of the equiangular spiral, with the characteristic angle β, whose value merely depends upon the two time-constants t_1 and t_2.

(4) Now imagine this length r constantly projected on some fixed straight line, and for simplicity take this straight line as at right angles to the direction in which $\theta = 0$.

Then the projection under consideration (E) has for its expression

$$E = a \cdot e^{-\frac{\theta}{\tan\beta}} \sin\theta;$$

i. e., it consists of a constant factor, a logarithmical factor, and a rhythmical or harmonic factor.

Substituting for θ its value $\frac{2\pi}{t_2} . t$,

$$E = a . e^{-\frac{2\pi}{t \tan\beta} . t} \sin \frac{2\pi}{t_2} . t,$$

$$E = a e^{-\frac{\theta}{\tan\beta}} \sin\theta,$$

$$\frac{dE}{d\theta} = a e^{-\frac{\theta}{\tan\beta}} \cos\theta - \frac{a}{\tan\beta} e^{-\frac{\theta}{\tan\beta}} \sin\theta$$

$$= a e^{-\frac{\theta}{\tan\beta}} \cos\theta - \frac{E}{\tan\beta}.$$

$$\frac{d^2E}{d\theta^2} = -\frac{1}{\tan\beta}\frac{dE}{d\theta} - a e^{-\frac{\theta}{\tan\beta}} \sin\theta - \frac{a}{\tan\beta} e^{-\frac{\theta}{\tan\beta}} \cos\theta$$

$$= -\frac{1}{\tan\beta}\frac{dE}{d\theta} - E - \frac{1}{\tan\beta}\left\{\frac{dE}{d\theta} + \frac{E}{\tan\beta}\right\}$$

$$= -\frac{2}{\tan\beta}\frac{dE}{d\theta} - E\left\{1 + \frac{1}{\tan^2\beta}\right\};$$

therefore

$$E + \frac{2\tan\beta}{1+\tan^2\beta}\frac{dE}{d\theta} + \frac{\tan^2\beta}{1+\tan^2\beta}\frac{d^2E}{d\theta^2} = 0.$$

Now

$$\frac{dE}{d\theta} = \frac{dE}{dt} \cdot \frac{dt}{d\theta} = \frac{dE}{dt} \cdot \frac{t_2}{2\pi},$$

therefore

$$\frac{d^2E}{d\theta^2} = \frac{d^2E}{dt^2} \cdot \frac{dt}{d\theta}\frac{t_2}{2\pi} = \frac{d^2E}{dt^2}\left(\frac{t_2}{2\pi}\right)^2,$$

therefore

$$E + \frac{2\tan\beta}{1+\tan^2\beta}\frac{t_2}{2\pi} \cdot \frac{dE}{dt} + \frac{\tan^2\beta}{1+\tan^2\beta}\left(\frac{t_2}{2\pi}\right)^2 \frac{d^2E}{dt^2} = 0;$$

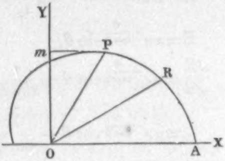
which, since $\tan\beta = \frac{2\pi t_1}{t_2}$, becomes

$$E + \frac{2t_1}{(1+\tan^2\beta)}\frac{dE}{dt} + \frac{t_1^2}{(1+\tan^2\beta)}\frac{d^2E}{dt^2} = 0;$$

and if t_3 is written for $\frac{t_1}{(1+\tan^2\beta)}$,

$$E + 2t_3\frac{dE}{dt} + t_1 t_3 \frac{d^2E}{dt^2} = 0.$$

An equation which is at once comparable with those obtained in the Electrical problem.



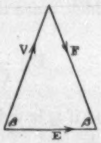
It thus appears that the variables in the problem of electrical discharge under consideration may be represented by the projections of three sides of a triangle, which is constantly undergoing uniform rotation and linear logarithmic shrinking. Let the figure represent a portion of the appropriate curve whose characteristic angle is β, and let O R be some radius vector. Then the projection of O R on O Y will be a maximum when the tangent at R is parallel to O X. Let P be such a position, and let P M be the tangent at P. Then $MPO = POX = \beta$.

Now suppose O A the line representing (in its projection) the effective electromotive force about to change sign through the value zero. This means that the current is about to change sign, and the condenser having been receiving current is about to begin to be discharged, *i. e.* its charge and therefore potential difference is a maximum. Then the line of P.D. must make the angle P O A with the line of *effective E.M.F.*

Again, the E.M.F. of self-induction is zero when the current is at a maximum, by the nature of the ordinary hypothesis.

Therefore, when the line representing E makes an angle β with O X, the line representing induced E.M.F. (F) must be parallel with it. Hence it also makes an angle β with the line of effective E.M.F.; but in phase lags behind it, whereas the P.D. is in advance by that angle.

Thus, if on any line taken as base we construct an *isosceles* triangle of appropriate base angles, the sides will represent the P.D. of the condenser and the induced E.M.F. of self-induction respectively, and the base will represent the effective E.M.F. It only remains to rotate the triangle with appropriate speed and to allow it to shrink at the due logarithmic rate.



The properties of the triangle agree exactly with the electrical properties.

The angle β is such that $\tan\beta = \dfrac{2\pi t_1}{t_2}$;

therefore
$$\cos^2\beta = \frac{1}{1+\left(\dfrac{2\pi t_1}{t_2}\right)^2} = \frac{t_3}{t_1} = \frac{\dfrac{KR}{2}}{\dfrac{2L}{R}} = \frac{KR^2}{4L}\,;$$

therefore $\cos\beta = \sqrt{\dfrac{K}{L}}\cdot\dfrac{R}{2}$, and β can have a real existence when $\dfrac{KR^2}{4L} < 1$;—the condition of Oscillatory Discharge.

The complete period is t_2, and is obtained in the electrical quantities thus :—
$$\tan\beta = \frac{2\pi t_1}{t_2}\,;$$
therefore
$$t_2^2 = \frac{(2\pi)^2 t_1^2}{\tan^2\beta} = \frac{(2\pi)^2 t_1^2}{\sec^2\beta - 1}\,;$$
and $\cos^2\beta = \dfrac{t_3}{t_1}$ already obtained ;

therefore
$$t_2^2 = \frac{4\pi^2 t_3 t_1}{1-\dfrac{t_3}{t_1}} = \frac{4\pi^2 KL}{1-\left(\dfrac{KR^2}{4L}\right)}\,;$$
therefore
$$t_2 = \frac{2\pi\sqrt{KL}}{\left(1-\left(\dfrac{KR^2}{4L}\right)\right)^{\frac{1}{2}}},$$
the form usually quoted if we neglect the second term of the denominator.

I purpose to show that in a discharge of the sort here contemplated (which has been shown to be the result of the ordinary premisses given at page 222) there will be no work done by any electromotive force which lags at an angle β behind the current, provided the initial condition is one of zero-current. And, further, that the source of E.M.F., which is represented by the side of the isosceles triangle in advance by the angle β, of the effective E.M.F., does all the work of heating the circuit and no more. It will thus be seen that there is *no provision in the theory for expenditure of power in the field*, and hence that the theory does not explain the well recognized phenomenon of radiation into space.

To establish the above-mentioned propositions, take the product of the projections of two lines undergoing variations corresponding to the two radii vectores of two equiangular spirals of the same characteristic angle β and period, and differing in phase by the angle 2γ.

One of these quantities may be expressed by

$$a e^{-\frac{\theta+\gamma}{\tan\beta}} \sin \overline{\theta+\gamma}.$$

The other by

$$b e^{-\frac{\overline{\theta-\gamma}}{\tan\beta}} \sin \overline{\theta-\gamma}.$$

The product is

$$ab \, . \, e^{-\frac{2\theta}{\tan\beta}} \sin \overline{\theta+\gamma} \, \sin \overline{\theta-\gamma},$$

or

$$ab \, . \, e^{-\frac{2\theta}{\tan\beta}} (\sin^2\theta - \sin^2\gamma).$$

This quantity, multiplied into an element of time dt, has to be integrated through one period. Since $\frac{d\theta}{dt} = \frac{2\pi}{t_2}$, the integral I becomes

$$\frac{abt_2}{4\pi}\int e^{-\frac{2\theta}{\tan\beta}} (\cos 2\gamma - \cos 2\theta)\, d\theta,$$

or

$$\frac{abt_2}{4\pi}\int e^{-\frac{2\theta}{\tan\beta}} \cos 2\gamma \, d\theta - \frac{abt_2}{4\pi}\int e^{-\frac{2\theta}{\tan\beta}} \cos 2\theta \, d\theta.$$

The first term of the integral is

$$-\frac{abt_2}{8\pi} \cos 2\gamma \tan\beta \, e^{-\frac{2\theta}{\tan\beta}}.$$

The second term is

$$\frac{abt_2}{8\pi} \sin\beta \cos \overline{\beta+2\theta} \, e^{-\frac{2\theta}{\tan\beta}};$$

and therefore the integral is expressed :—

$$\mathrm{I} = \frac{abt_2}{8\pi} e^{-\frac{2\theta}{\tan\beta}} \{\sin\beta \cos \overline{\beta+2\theta} - \tan\beta \cos 2\gamma\}.$$

This expression has to be taken between limits. If we contemplate one revolution only the limits will be $\Theta_1+2\pi$ and θ_1, and the Definite Integral becomes

$$\frac{abt_2}{8\pi}\{-\sin\beta\cos\overline{\beta+2\theta_1}+\tan\beta\cos 2\gamma\}e^{-\frac{2\theta_1}{\tan\beta}}\left(1-e^{-\frac{4\pi}{\tan\beta}}\right).$$

If the limits are infinity and θ_1 the integral becomes

$$\frac{abt_2}{8\pi}\{\tan\beta\cos 2\gamma-\sin\beta\cos\overline{\beta+2\theta_1}\}e^{-\frac{2\theta_1}{\tan\beta}}. \quad . \quad (\alpha)$$

Either of these expressions becomes zero when

$$\tan\beta\cos 2\gamma=\sin\beta\cos\overline{\beta+2\theta_1}$$

or

$$\cos 2\gamma=\cos\beta\cos\overline{\beta+2\theta_1},$$

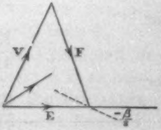
showing that the condition that no work shall be done in the electric problem depends on the initial circumstances, *i.e.* θ_1 is involved. If $2\gamma=\beta$ the condition of no work becomes

$$\cos\overline{\beta+2\theta_1}=1,$$

which is satisfied when

$$\theta_1=-\frac{\beta}{2}.$$

Hence if the initial condition be that of no current, the line bisecting the angle between the line of effective E.M.F. and that of the self-induced E.M.F. makes $-\frac{\beta}{2}$ with the line $\theta=0$, and it is thus proved that on the whole no work is done in the field.



If, on the other hand, we make

$$\theta_1=+\frac{\beta}{2},$$

and start from a point where the current is zero, we have in the above expression, when proper substitutions are made for a and b, the value of the work done on the circuit by the discharging condenser. The integral between infinity and θ_1 becomes, when $\theta_1=\frac{\beta}{2}$ and $2\gamma=\beta$,

$$\frac{ab\,.\,t_2}{8\pi}\{1-\cos 2\beta\}\sin\beta\,.\,e^{-\frac{\beta}{\tan\beta}},$$

or

$$\frac{ab \, . \, t_2}{4\pi} \sin^3 \beta e^{-\frac{\beta}{\tan \beta}}.$$

In this case $b = \frac{E}{R}$ and $ae^{-\frac{\beta}{\tan\beta}} \sin \beta$ is the potential difference between the plates of the condenser at starting $= V_1$, say, $= V \sin \beta$. Hence the expression becomes

$$\frac{E}{R} \cdot \frac{t_2}{4\pi} \sin^3 \beta \, . \, V$$

(E and V being now the full sides of the triangle, properly interpreted), and $\frac{E}{2 \, . \, V} = \cos \beta$ by the geometry of the triangle; and further,

$$\tan \beta = \frac{2\pi t_1}{t_2} \text{ and therefore } \frac{t_2}{4\pi} = \frac{t_1}{2 \tan \beta},$$

and the work

$$= \frac{E^2}{R} \frac{1}{2 \cos \beta} \frac{t_1}{2 \tan \beta} \sin^3 \beta, \text{ or } \frac{E^2}{R} \frac{t_1 \sin^2 \beta}{4},$$

which is the expression we should obtain if we integrate the square of the current multiplied by R*dt*, seen as follows:—

In the general expression (α) obtained above for the product of the projections make $a = E$, $b = \frac{E}{R}$, and $\theta_1 = 0$, $\gamma = 0$, the expression (α) becomes

$$\frac{E^2 t_2}{8\pi R} \{\tan \beta - \sin \beta \cos \beta\},$$

or

$$\frac{E^2}{4R} \cdot \frac{t_1}{\tan \beta} \{\tan \beta - \sin \beta \cos \beta\},$$

or

$$\frac{E^2}{4R} t_1 \sin^2 \beta, \text{ as above.}$$

Thus the whole of the work goes to heat the wire, and, further, substituting in the equation for E in terms of V, it may be shown to be entirely derived from the charged condenser.

The work may be written, eliminating E,

$$\frac{2V^2 \cos \beta}{R} \frac{t_2}{4\pi} \sin^3 \beta,$$

or

$$\frac{V^2}{R}\frac{t_1}{\tan\beta}\cos\beta\sin^3\beta.$$

Now $V_1^2 = V^2\sin^2\beta$, and thus the work is

$$V_1^2\frac{t_1}{R}\cos^2\beta,$$

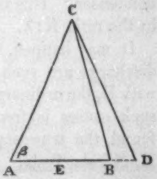
or, since $\cos^2\beta = \frac{t_3}{t_1}$,

$$= \frac{V_1^2 t_3}{R} \text{ and } t_3 = \frac{KR}{2}$$

$$= \frac{V_1^2 K}{2},$$

which is the ordinary expression for the energy stored in the condenser; and this appears from the investigation to be entirely expended in heating the circuit, and there is no margin for the exhibition of power elsewhere.

Suppose a line A B to represent (E) the line of effective E.M.F. At the extremity A set off A C as the direction of the line representing the P.D. of the condenser.



Then, as the condenser contains all the energy that is going to be expended on the circuit and on the æther, from what has been said it is clear that A C must be rather longer than the side of the isosceles triangle; for, if not, the energy stored will not do more than heat the circuit. If, therefore, a perpendicular be dropped upon A B from C, it will fall at a point nearer B than A.

Join C B, and, further, draw C D to meet A B produced in D, and so that C D A is an isosceles triangle on A D as base, and therefore C D A $=\beta$. Now C B must be the line representing the resultant of the induced electromotive forces F, and however complicated the case may be this line C B is equivalent to two components C D, D B; of which C D results in no expenditure of power because it is in a phase β *behind* the current, and D B is in phase directly opposed to the current, and therefore resulting in whatever expenditure of energy takes place outside the circuit, and therefore in the æther or in magnetic bodies, or in neighbouring or surrounding conductors. As in the former case of sustained oscillations, it may be shown that B C D is a *magnetic lag* necessary for the exhibition of such phenomena.

The electromotive force DB may be expressed by $-\lambda C$ as before, and the general equation

$$V+F=E$$

takes the form

$$V-L\frac{dC}{dt}-\lambda C=RC,$$

and, as this may be written

$$V-L\frac{dC}{dt}=(R+\lambda)C,$$

we see that the extra consideration required to express the actual state of things is simply that the resistance of the circuit is virtually increased. In the previous work it is necessary to write $(R+\lambda)$ in all the equations.

The actual work done altogether is derived from the charged condenser. This is divided between the circuit and the field in the ratio $R:\lambda$.

It may happen, therefore, that if the circumstances of the discharge are such as to make λ very large in comparison with R, the ordinary heating-effect may be minimized. Among such causes is frequency, and in this consideration is to be found the true explanation of some of the experiments of M. Nikola Tesla. The energy of the discharges which that physicist encountered was expended in chief part in radiation which his body did not check, and not in current through his body. It is here suggested that the best way to measure radiation would be to measure the *defect* in the heating of a circuit, taking care to note the P.D. of the condenser at the moment previous to discharge.

In ordinary *sustained* oscillations, as derived from a machine, the alternations are not of sufficient frequency to make the effect of λ perceptible. Electromotive forces of induction involve the period in their denominators, and it is reasonable to suppose that induced magnetic fields do the same; and if the period of the electromagnetic vibrations becomes comparable with that of light, it is conceivable that mere heating might vanish, as in the solar spectrum light has less heating-effect than radiation of smaller frequency from the same source.

DISCUSSION.

Prof. Perry thought the C^2R term would not represent the heating of the wire when the oscillations were rapid, owing to the distribution of current not being uniform over the section of the conductor. Maxwell had shown that certain throttling terms had to be considered. In condenser discharges the complete equation would have many terms.

Prof. O. J. Lodge said the best definition of R in such cases was that derived from Joule's law rather than that of Ohm. Frequency was very important in the radiation of energy, but even at ordinary frequencies of alternators some energy was radiated. Referring to Tesla's experiments, he said the reason why no serious consequences followed was that there was not much energy behind them. High frequency might be instrumental in preventing injury, but this he thought remained to be proved.

Dr. Sumpner pointed out that losses other than C^2R (R being the ordinary resistance of the conductor) had to be taken into account. In some cases, such as transformers on open circuit, the effective resistance might be one thousand times that of the coil. To discuss completely the problem taken up by Mr. Blakesley, it would be necessary to take account of non-uniform distribution of current both across and along the conductor, as well as the character of the magnetic and electric fields surrounding the circuit.

Mr. Swinburne thought there was a tendency to overestimate the rate of high-frequency currents, for unless the coils of transformers were assumed geometrically coincident, calculations were difficult. Errors of hundreds per cent were quite possible. In Tesla's experiments no great power was involved, for the transformer could not give out any large power.

Mr. Blakesley, in reply, said the term R was such that C^2R represented the whole waste in the conductor, whilst $C^2\lambda$ included everything wasted outside the conductor.

XV. *Liquid Friction.* *By* JOHN PERRY, *F.R.S., assisted by* J. GRAHAM, *B.A., and* C. W. HEATH*.

[Plate II.]

A PIECE of apparatus such as is used in this investigation was designed and partly constructed in Japan in 1876; it is described in my book on Practical Mechanics (1883). The specimen actually used by us was constructed at the Finsbury Technical College in 1882, and has been occasionally used since that time, but no complete sets of observations were attempted till October 1891.

The simplest hydrodynamical condition of viscous fluid is that of the fluid bounded by two infinite parallel planes, the fluid in one boundary being at rest, the velocity in the other boundary being constant and in the plane. Motions in a pipe and near a vibrating disk, or even near a steadily rotating disk, are rather complicated. Our apparatus was designed to approach as nearly as possible to the conditions subsisting between the infinite planes. Between two such planes, if V is the constant velocity of one of them, the other being at rest, and b is their distance asunder, the fluid being of uniform density, and gravity being neglected, if z is measured at right angles from the fixed plane, the equation of steady motion is

$$\frac{d^2v}{dz^2} = 0, \quad \dots \dots \dots \quad (1)$$

or

$$v = \mathrm{V}z/b \; ; \quad \dots \dots \dots \quad (2)$$

and the tractive force per unit area required at the moving plane to maintain the motion is $\mu\mathrm{V}/b$, where μ is the coefficient of viscosity. We have used, instead of planes, concentric cylindric surfaces of as large radii and as small difference in radii as could be conveniently constructed and used (see Pl. II. fig. 2).

E E E is a cylindric trough, of which the curved parts E and E are brass. The inner and outer radii of this trough are 10·39 and 12·65 centimetres. C, which forms the bottom, is of iron; and the whole trough can be rotated about its

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* Read March 24, 1893.

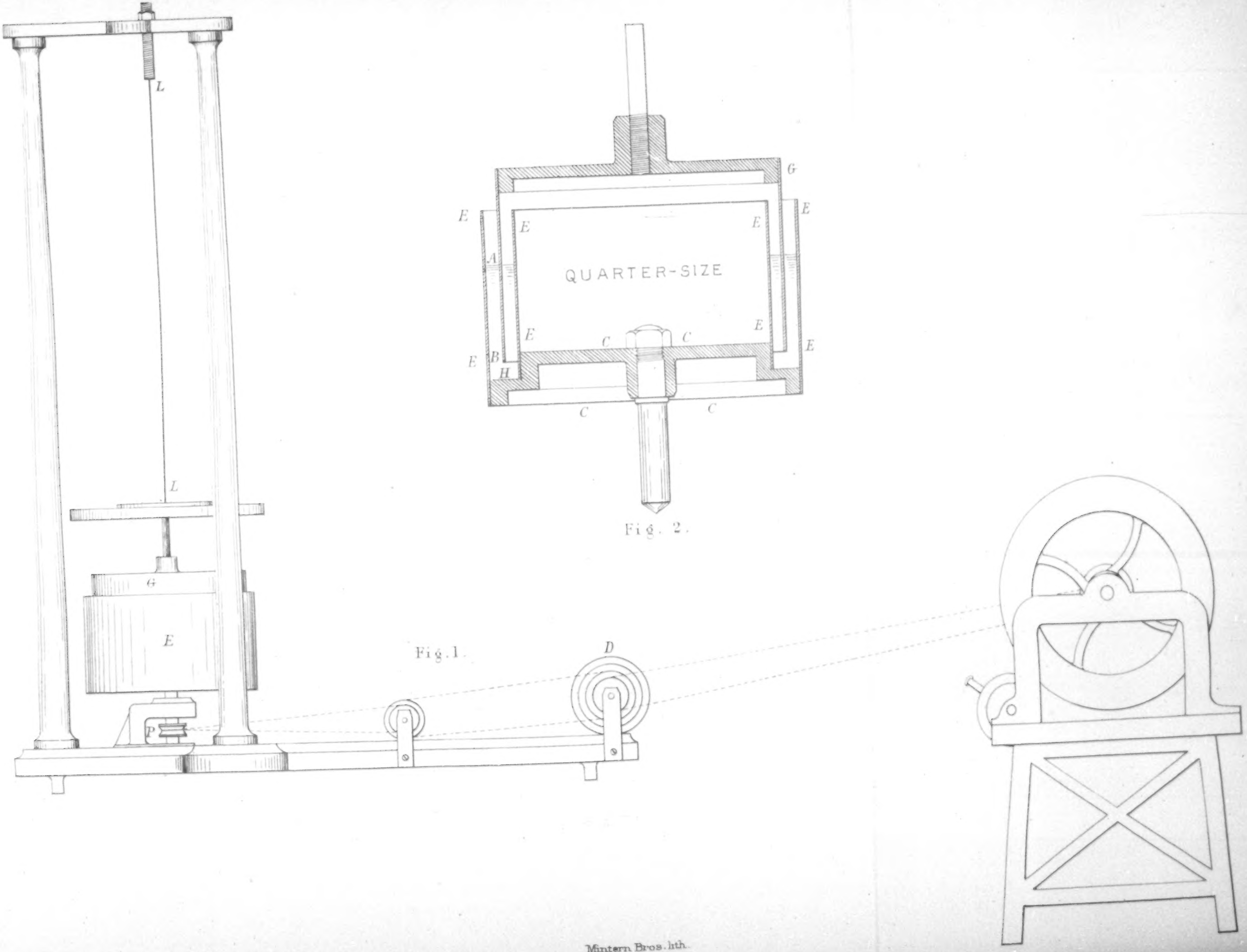

Fig. 1.

Fig. 2.

Mintern Bros. lith.

vertical axis at any desired speed by driving the pulley P from a coned pulley D with numerous steps.

G is a hollow brass cylinder supported by a steel wire L, of 0·037 centim. diameter, 67·78 centim. long, whose axis coincides with the axis of the trough and the axis of rotation. G may be raised or lowered relatively to the trough. The outer radius of G is 11·63 centim., the inner being 11·41 centim. The whole apparatus is supported on a stand, with three adjustable feet. We exhibit also some photographs of the apparatus in position, showing how it was driven.

The trough contains the liquid whose viscosity is to be measured: when it rotates, G tends to rotate; and when for any constant speed G is in equilibrium, the twist in the steel wire measures the torque due to the tractive forces with which the liquid acts upon G at its inner and outer surfaces. The twist was measured by the angular motion of a pointer clamped on the wire at a distance of 59 centim. from the fixed end.

To test the accuracy of our assumption that the fluid behaved as if between parallel plane surfaces, let us consider the actual motion in which the stream-lines are circles. Consider the motion of a stream-tube of section $\delta r\,\delta x$, x being measured axially and r radially. The tangential force on unit cylindric surface of radius r is $\mu\left(\frac{dv}{dr}-\frac{v}{r}\right)$, if v is the velocity. The moment due to all such forces on the inner surface of our ring is

$$2\pi r^2 . \mu\left(\frac{dv}{dr}-\frac{v}{r}\right)\delta x.$$

The moment tending to increase the velocity of the ring due to forces on the cylindric parts of it is therefore

$$2\pi\mu . r^2\left(\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}-\frac{v}{r^2}\right)\delta r . \delta x;$$

also, due to the plane faces we have the moment

$$2\pi\mu r^2\frac{d^2v}{dx^2}\delta r . \delta x.$$

Equating the sum of these to the rate of increase of the moment of momentum of the ring, we have

$$\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}-\frac{v}{r^2}+\frac{d^2v}{dx^2}=\frac{\rho}{u}\frac{dv}{dt} \quad . \quad . \quad . \quad . \qquad (3)$$

as the equation of motion in co-axial circular stream-lines.

Now the discontinuity at the edge, and also the nearness of the bottom of the trough, cause the term $\frac{d^2v}{dx^2}$ to be important; but the solution seems to be very difficult. Maxwell satisfied himself (Collected Papers, vol. ii. pp. 16–18) that the discontinuity at the edge of a vibrating disk could be allowed for as a virtual increase in the radius of his disk, and the assumption that the behaviour of his fluid was the same as if his disk were part of an infinite disk. The correction not being readily obtained for a disk, he assumed it to be the same as for the straight edge of an infinite plane surface. We are certainly not less correct in taking the same correction for the edge of our cylinder *. Following Maxwell, therefore, we assumed that when our cylinder G was immersed to the depth AB or l in the fluid it was really a portion of length $l+\lambda$ of an infinite cylinder of the same diameter. We therefore neglect $\frac{d^2v}{dx^2}$ in (3), and we use

$$\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}-\frac{v}{r^2}=\frac{\rho}{\mu}\frac{dv}{dt}. \quad . \quad . \quad . \quad . \qquad (4)$$

When the motion is steady, that is when $dv/dt=0$, the solution is

$$v=\mathrm{A}r+\mathrm{B}/r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

If $v=v_1$ when $r=\mathrm{R}_1$, and $v=0$ when $\mathrm{R}=\mathrm{R}_2$, then

$$v=v_1\mathrm{R}_1(r-\mathrm{R}_2{}^2/r)/(\mathrm{R}_1{}^2-\mathrm{R}_2{}^2).$$

We must now distinguish between the space outside the

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* It is to be remarked that Maxwell assumed, generally, that there was no radial motion of his fluid. Now there must have been radial motion, his disks resembling centrifugal fans in their action, creating a variable flow always outwards between his fixed and moving disks; and the energy wasted in producing this flow is neglected by him. We do not know the amount of this error, and he may have satisfied himself as to its insignificance. Prof. Maurice FitzGerald in criticizing this proof has pointed out the fact that on James Thomson's theory of river bends there must exist a radial motion of an interesting kind in our apparatus.

suspended cylinder and the space inside it. The radii of the inner and outer surfaces of the suspended cylinder are 11·41 and 11·63 centim., and the inner and outer radii of the trough are 10·39 and 12·65 centim.

Our cylindric surfaces were not perfectly true, although great care was taken to make them so; and the radii given are only average dimensions. But, inasmuch as slightly tilting the apparatus or otherwise putting the axis of the suspended cylinder out of coincidence with the axis of the trough made only small differences in the observations, we did not think that such inaccuracies of workmanship or measurement as existed could affect our results.

Even when the tilting of the apparatus was quite evident to the eye, the tractive torque was found to be only slightly increased by the tilting. Of course, as the suspended cylinder got closer and closer to the side of the trough the torque did increase, and became very large when the suspended cylinder nearly touched the side of the trough.

Again, it was observed that at our highest speeds the amount of wetted surface did not perceptibly alter; and we are, we think, justified in assuming that the surface of the liquid was always a plane surface.

It is evident that the tractive forces on the suspended cylinder are the same whether we assume the trough to revolve steadily at ω radians per second, the suspended cylinder being at rest, or the suspended cylinder to revolve steadily at ω radians per second and the trough to remain at rest. We shall therefore, for ease of calculation, always assume the trough to be at rest and the suspended cylinder to be revolving at ω radians per second. Then the velocities of its inner and outer surfaces are, in centimetres per second, $11{\cdot}41\,\omega$ and $11{\cdot}63\,\omega$.

On any cylindric surface the tractive force per unit area being $\mu\left(\frac{dv}{dr}-\frac{v}{r}\right)$ is $-\frac{2B}{r^2}\mu$ from (5); so that, whether for the outer or inner space, if R_1 is the radius of the suspended moving cylindric surface, and R_2 the radius of the fixed surface, the tractive moment per centim. of length is

$$\pm 4\pi\omega\mu R_1^2/(R_1^2/R_2^2-1).$$

Taking actual sizes, this is 0·5 per cent. greater than the value obtained by calculating the forces on the assumption that the fluid moves in plane layers as in (2), b being the actual thickness of fluid 1·02 centim., and V being the actual velocity at the mean radius. We may, in fact, imagine the speeds to be increased by 0·5 per cent., and make all calculations as to viscosity on the assumption of motion in plane layers.

The tractive torque per centimetre of length of cylinder is, in our case, $19010\mu\omega$, or $1991 n\mu$ if the angular velocity is given as n turns per minute. If l is the wetted length in centimetres, and λ is the virtual additional length representing the edge effect, the total torque is $1991 n\mu(l+\lambda)$. The total observed motion of the pointer being D degrees, and the torque per degree being a, the torque due to tractive forces acting on the cylinder is

$$a\mathrm{D} = 1991\, n\mu(l+\lambda)\ ;$$

and if this law is found to be true experimentally, then

$$\mu = a\mathrm{D}/\{1991(l+\lambda)n\}. \quad . \quad . \quad . \quad . \quad (6)$$

Two methods of determining the torsional constant of the wire were employed :—

First Method.—A fine cotton thread was wound round the outside of the suspended cylinder and passed over a nearly frictionless pulley (the pulley of an Attwood's machine) to a scale-pan. The thread was nearly horizontal as it left the cylinder. In this way it was found that the twisting moment required to produce a pointer-rotation of one degree was 1531 dyne-centimetres. In making the measurement, as the weight of the scale-pan and its contents was gradually increased, the steel wire was drawn away from the vertical, and therefore from the middle of the scale; but the stand was tilted to counteract this effect.

The effects due to solid friction were eliminated by taking the mean of the limiting weights for equilibrium. When the weight was 30 grams, one tenth of a gram either added to or taken from the scale-pan produced a perceptible change in the position of the pointer ; so that the solid friction was small.

Second Method.—The suspended cylinder was allowed to vibrate, twisting and untwisting the wire; and its times of oscillation were noted. The observations were repeated when a known moment of inertia had been added. Unloaded, it made 40 complete oscillations in 583 seconds, or one oscillation in 14·575 seconds. We then attached to the cylinder an iron bar of rectangular section, whose own moment of inertia had been determined accurately by previous experiments (found to agree with calculation on the assumption that it was homogeneous), this moment of inertia being 566·2 (in gram-centimetre^2 units). The time of a complete oscillation was now found to be 21·425 seconds. It follows that the moment of inertia of the suspended cylinder is 487·72, and the torsional constant of the wire is readily obtained. This constant being corrected on account of the position of the pointer, it follows that to produce a rotation of the pointer of one degree requires a torque of 1552 dyne-centimetres. This is greater than the constant derived from direct measurement by 1⅓ per cent.; but, on the whole, we are rather inclined to accept the number obtained directly, as we are not quite sure that the mean position of the iron bar was at right angles to the magnetic meridian.

Eight quite independent measurements of the diameter of the wire were made by men experienced in making such measurements; and the mean value was ·0371 inch, the greatest and least being ·0373 and ·0369. Using this mean value, and the directly measured torsional constant, it would seem that the modulus of rigidity of the steel is $7·71 \times 10^{11}$. As an error of ·0004 inch in the diameter measurement leads to an error of 4 per cent. in the modulus of rigidity, and as the modulus of rigidity usually taken for steel is $8·19 \times 10^{11}$, we believe that our constant 1531, as directly measured, is sufficiently correct for practical purposes.

In using (6), then, we take a to be 1531.

The virtual addition λ ought, by Maxwell's formula, to be 0·52 centim. in our case. But the bottom of the trough H was only 0·5 centim. from the edge B of the suspended cylinder in most of our experiments, and we do not know how to calculate for this. Our experiments have shown that when

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this distance is 0·5 centim. the twisting moment at a given speed is practically the same as when the distance is much greater; but we did not know this from any theory, and, besides, it is always rather dangerous to depend upon a theoretical calculation of λ such as Maxwell was compelled to use. It is possible, also, that a correction of the same kind ought to be introduced for capillary and other actions at the surface of the liquid. The action of the atmosphere was in any case negligible, because when there was no liquid present in the trough, so that there was an action of the air several times greater than ever occurred during the experiments, the deflexion was quite imperceptible at much higher speeds than those used in the experiments.

The temperature being kept as nearly as possible constant, but probably varying between 18°·9 C. and 20°·1 C. (stated as 19°·5 C.), the following experiments were made with sperm-oil, beginning with a small quantity in the trough and ending with a large quantity. The bottom of the trough was in every case 0·5 centim. below the edge of the suspended cylinder.

TABLE I.—June 9th, 1892.

n.	Deflexion D when				
	l=0·5 cm.	l=2·5 cm.	l=5 cm.	l=7·5 cm.	l=9·9 cm.
50	24	67	112	158	
49	...	...	...	...	196
48					
40	18	...	...	132	163
39	...	53	91		
23	10				
17·5	8	..	...	57	73·5
17	...	24	39		
12	5	...	...	40	
11·5	...	16	27		
11	..	...	...	...	49

As the deflexion is sufficiently nearly proportional to the speed to allow of corrections by this rule when the corrections are small, we have corrected the above observations to the speeds 50, 40, 17½, 11½; and we obtain the following results:—

Values of l.	Values of n.			
	50.	40.	17½.	11½.
·5	24	18	8	5
2·5	67	54	25	16
5·0	112	93	40	27
7·5	165	132	57	38½
9·9	200	163	73·5	51

D and l were then plotted as the coordinates of points on squared paper; and it was obvious that for each value of n these points lay very nearly in a straight line, and all the straight lines passed through the point $l=-0{\cdot}8$. It is curious that the linear law should hold for such small values of l as 0·5 centim., and for high speeds as well as low speeds. We shall presently see that some of these speeds are considerably above the critical speed at which (4) ceases to represent the motion.

We may take it, then, that $\lambda=0{\cdot}8$ centim., which is greater than the calculated value 0·52. The discrepance cannot be due to the distance BH being small, for we have altered this distance and found no perceptibly different results. As already stated, it may be due to some capillary surface action. Taking $a=1531$ and $\lambda=0{\cdot}8$, we have (6) becoming

$$\mu=0{\cdot}769\,\mathrm{D}/n(l+0{\cdot}8). \quad . \quad . \quad . \qquad (7)$$

Of course our results are consistent with our equations of motion only so long as $\mathrm{D}/(l+0{\cdot}8)$ is proportional to n.

Many observations have been made with this apparatus during the last year on various liquids, under very different conditions of temperature and speed and depth. We give here a set made on sperm-oil. In all cases the bottom of the trough was 0·5 centim. below the edge of the suspended cylinder.

Keeping the oil at a constant temperature we ran the trough at a number of speeds, and repeated at other constant temperatures. The results are given in Tables IV. to XI.

When a temperature had to be taken the rotation was stopped and a thermometer dipped about halfway down in the oil, the reading being taken at the end of about half a minute. A small Bunsen flame was applied underneath the trough when a temperature higher than the room had to be maintained for a considerable length of time.

As the temperature varied slightly, and we wished to reduce our observations to constant temperatures, we afterwards made two sets of observations at very varying temperatures but constant speeds. These later observations we shall consider first. They are given in Tables II. and III.

TABLE II.—March 29th, 1892. ($l=8{\cdot}275$ centim.)

n.	$\theta°$ C.	D.	$\frac{40}{n}\frac{D}{l+\lambda}$.	μ.	μ calculated.
39·5	13·5	189	21·09	·400	·446
39·5	17·1	165	17·60	·334	·357
40	19·5	150	16·53	·318	·317
40	24·0	134	14·77	·287	·266
40	25·3	126·5	13·94	·268	·254
40	28·9	111·5	12·28	·236	·228
40	32·0	109	11·46		
40	42·5	88	9·70		
40	46·9	81	8·65		
40	58·5	67	7·11		
40	64·0	58	6·39		
40	71·0	56	6·17		
40	77·0	50	5·51		
40	85·5	46·5	5·12		

The numbers in the column headed $\frac{40}{n}\frac{D}{l+\lambda}$ are obviously intended to be corrections of $D/(l+\lambda)$ for the constant speed of 40 revolutions per minute.

TABLE III.—March 30th, 1892. ($l=7{\cdot}78$ centim.)

n.	$\theta°$ C.	D.	$9\frac{D}{l+\lambda}$.	μ.	μ calculated.
8·75	5·2	160	19·19	1·64	2·06
9·75	8·0	85	9·15	0·78	0·82
9·2	10·0	57	6·56	0·73	0·62
9·0	10·8	47	5·48	0·47	0·56
9·0	16·6	36·5	4·26	0·363	0·366
9·2	24·8	26·5	3·02	0·258	0·259
9·0	35·0	19·5	2·27	0·194	0·196
9·0	47·0	14·0	1·63	0·139	0·137
9·0	56·5	10·2	1·19	0·101	0·104
9·0	67·0	8·0	0·93	0·079	0·081
9·0	89·0	5·4	0·63	0·060	0·058
9·0	84·5	6·0	0·70	0·054	0·054

The numbers headed $\frac{9}{n}\frac{D}{l+\lambda}$ are intended to be corrections of $D/(l+\lambda)$ for the constant speed of 9 revolutions per minute.

We have plotted the numbers in the last columns of these tables with θ upon a sheet of squared paper; but it is unnecessary to publish the resulting curves. We exhibit them to the Members of the Society.

Knowing what has been done by Prof. Osborne Reynolds, it seemed unlikely that one simple formula should satisfy either of these curves; that is, it was likely that in the lower curve there was some temperature for which the speed $n=9$ was a critical speed, and there was also a temperature for which $n=40$ was a critical speed. We therefore used the curves merely for small temperature-corrections in our other experiments, in which we kept the temperature nearly constant.

It was therefore without much interest that, in preparing this paper for publication, we tried to obtain empirical formulæ for these curves; and at first we used, not the observations themselves, but the observations as corrected by curves drawn upon squared paper.

When $\log(\theta-4{\cdot}2)$ and $\log\frac{D}{l+\lambda}$ are plotted as coordinates of points on squared paper, we were astonished to find that when $n=9$ the points lie in two straight lines. The allineation of the points is very striking, even when the uncorrected observations are taken, and leads to the following empirical formula:—

Letting ϕ denote $\theta-4{\cdot}2$, and letting y denote $D/(l+\lambda)$, the torque as measured in degrees deflexion of pointer per unit length of wetted cylinder; then, at the constant speed of $n=9$,

$$y\phi^{1\cdot 349}=\text{constant for temperatures above } 40^\circ\text{ C.}$$

$$y\phi^{0\cdot 686}=\text{constant for temperatures below } 40^\circ\text{ C.}$$

On plotting $\log y$ and $\log\phi$ for the constant speed $n=40$, the points are not found to lie so nicely in straight lines, but there does seem to be some sort of discontinuity at a temperature of about 45° C.

At first we thought that these temperatures were the temperatures at which the speeds $n=9$ and $n=40$ were the critical speeds, and we were greatly concerned because our result seemed to be quite out of accord with the reasoning of Prof. Osborne Reynolds *. As his ingenious theory has been completely verified by experiments made upon the very smallest and largest pipes with flowing water, and as it is simple we had adopted it for the reduction of our experiments.

According to his theory, $\frac{D}{l+\lambda}$ or, as we shall call it, y, ought to be proportional to n until n exceeds a certain value; this value being a function of μ/ρ, where ρ is the density of the fluid. Now the alteration of ρ with temperature in such a liquid as sperm-oil is so small that the error in neglecting it is small in comparison with our errors of experiment.

Neglecting, then, the alterations in ρ, the theory of Prof. Reynolds leads to

$$y=a\text{F}^{2-\kappa}n^{\kappa}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where F is a function of the temperature, n the number of revolutions per minute; where $\kappa=1$ until the critical speed n_c is reached, n_c being proportional to F, and κ having a higher value than 1 for all speeds above the critical; a is a constant. This is on the assumption that Prof. Reynolds's theory would lead to the same result in our case as in his pipes.

Now, in the first place, it seemed absurd that the temperatures for which the speeds 9 and 40 were the critical speeds should be so near to one another as 40° C. and 45° C. But a much more serious consideration was this. According to any reasonable application of the theory to our case, at constant speed, if $y\phi^m$ is constant when the speed is less than the critical speed, and if $y\phi^s$ is constant when the speed is above the critical speed, then s ought to be less than m, whereas 1·349 is about twice 0·686. We came to the conclusion that the point of discontinuity has nothing whatever to do with the critical speed; indeed, we subsequently found

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* "An Experimental Investigation of the Circumstances which determine whether the Motion of Water shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels," by Osborne Reynolds, F.R.S., Phil. Trans. pt. iii. (1883).

it probable that $n=9$ does not become the critical speed until the highest temperature of Table III. is reached.

Using the deflexions in Table III. to determine μ according to (7), we have the results given in column 5 of the Tables. The numbers in column 5 of Table I. are calculated for temperatures lower than 26° C., which is about the temperature at which 40 is the critical speed. In some of the following tables, giving the results of experiments made at various constant temperatures, we have also given values of μ. There is as much consistency in all these results as might have been expected. We lay most weight upon the results given in Table III., which lead to the laws

$$\mu = 2{\cdot}06(\theta-4{\cdot}2)^{-{\cdot}686} \text{ below } 40^\circ \text{ C.}; \quad . \quad . \quad . \quad (9)$$

$$\mu = 21{\cdot}67(\theta-4{\cdot}2)^{-1{\cdot}349} \text{ above } 40^\circ \text{ C.} \quad . \quad . \quad . \quad (10)$$

We have searched in books in vain for a mention of a discontinuity in any other physical property of sperm-oil about this temperature; but we have already begun to experiment on its other physical properties, as it is unlikely that there should be a discontinuity in the law for the viscosity alone. At the same time, we may say that our chemical friends see no reason for a confirmation of our belief.

In the tables we give the viscosity as calculated from these formulæ; and it will be seen that they agree well enough with the observed viscosities.

TABLE IV.—March 18th, 1892.

($l=6{\cdot}15$ centim. Temperature Constant, 17°·5 C.)

n.	D.	y, or D÷$(l+\lambda)$.	μ.
36	114	16·41	·351
39	121	17·41	·344
54	172	24·75	·352
69	245	35·26	
80	300	43·16	
92	345	49·64	
23	74	10·65	·356
16	52	7·48	·360
13	41	5·90	·349
8·75	27	3·98	·349

The column headed μ is $0{\cdot}769\,\text{D}/n(l+\lambda)$, and has no meaning at a speed greater than the critical speed. The critical speed, n_c, is probably about 50. The first three and first four values are probably measurements of μ. The average value of these seven is $\mu = 0{\cdot}351$. Formula (9) would make μ to be 0·349.

Plotting $\log y$ and $\log n$ as the coordinates of points on squared paper, the points lie very nearly in a straight line indicating $y \propto n$ until the critical speed, about $n = 50$, is reached, and for all higher speeds the points lie nearly in another straight line indicating $y \propto n^{1{\cdot}25}$.

TABLE V.—March 21st, 1892. ($l = 6{\cdot}075$ centim.)

θ° C.	n.	D.	y.	μ.
22·5	9·25	24·5	3·40	·283
23·2	11	28·0	3·97	·278
22·7	14·2	37	5·16	·280
22·0	17·2	45	6·15	·279
25·0	27	68	10·19	·277
24·5	23	57	8·41	·281
24·0	32	77	11·20	·269
23·0	90	340	48·00	
22·5	102	410	56·96	
23·0	80	276	38·94	
24·0	72	224	32·58	
24·5	56	169	24·95	
24·0	48	129	18·76	
23·5	43	108	15·42	·278
23·0	38	97	13·69	·277

The column headed y is $D/(l + \lambda)$ corrected to the constant temperature of 24° C. by a correction of about 3 per cent. per degree. The numbers in the last column have no meaning for speeds higher than $n =$ about 43. The average value of μ, the viscosity, in the first seven and last two observations is 0·278. Formula (9) would make μ for this temperature 0·266.

Plotting $\log n$ and $\log y$ on squared paper gives points lying nearly in two straight lines, showing that $y \propto n$ to the critical speed $n =$ about 43, and above that speed $y \propto n^{1{\cdot}37}$.

TABLE VI.—March 21st, 1892. ($l=6·075$ centim.)

$\theta°$ C.	n.	D.	y.
29·5	38	81	11·63
30	43	95	13·82
31	48	114	16·99
30·5	56	134	19·73
30·5	60	147	21·65
30	66	175	25·45

y here means $D/(l+\lambda)$ corrected to the constant temperature of 30° C. by a correction of $2\frac{1}{2}$ per cent. per degree. μ as calculated from (7) would have no meaning, as the critical speed for this temperature is about 28 revolutions per minute, and we give no column headed μ. y is very nearly $\propto n^{1·33}$.

TABLE VII.—March 22nd, 1892. ($l=5·425$ centim.)

$\theta°$ C.	n.	D.	y.	μ.
56	38	49·5	7·92	
55	42	59	9·20	
55	54	72·5	11·31	
56	58	88	14·13	
56·5	74	116	18·79	
57·5	108	158	26·32	
58·5	29	34	5·78	
56	24	25	4·01	·1282
57	17·5	11	1·81	·0791
57	15·5	14	2·31	·1143
54	15·5	12	1·82	·0903
55	16	14	2·19	·1054

y means $D/(l+\lambda)$ corrected to 56° C. by a correction of $2\frac{1}{2}$ per cent. per degree. The critical speed is probably below $n=24$, and for speeds greater than the critical $y \propto n^{1·32}$ nearly. For speeds less than this the average value of μ is 0·103. According to (9) the value of μ for 56° C. is 0·1055.

Plotting $\log y$ and $\log n$ as the coordinates of points on squared paper gives points which may be said to lie on two straight lines, but the errors of observation are too great. For $n>24$ we might perhaps say that $y \propto n^{1·28}$; but it seems hardly fair to draw conclusions from this set of observations.

TABLE VIII.—March 24th, 1892. ($l=7·025$ centim.)

$\theta°$ C.	n.	D.	y.	μ.
30	100	363	45·24	
30·8	78	250	31·79	
31	115	404	51·64	
31	32	62	7·92	·1903
31	22	47	6·01	·2103
31	17	37	4·73	·214
32	13	29	3·80	·2249
31	11	23·5	3·00	·2097
30	9·2	20	2·50	·209

y is D/$(l+\lambda)$ corrected to 31° C. by a correction of 2½ per cent. per degree. The numbers of the last column have no meaning for speeds higher than n about$=32$. The average value of μ is 0·210. According to (9) the value of μ for 31° C. is 0·216.

Below $n=32$, $y \propto n$. Above $n=32$, we may perhaps say that $y \propto n^{1·4}$.

TABLE IX.—March 28th, 1892. ($l=6·525$ centim.)

$\theta°$ C.	n.	D.	y.
81	38·5	38	5·18
80·5	56	63	8·557
80	69	88	11·80
79·5	84	112	14·70
—	104	134	18·29
83	92	116	16·157

y means D/$(l+\lambda)$ corrected to 81° C.

The law seems to be $y \propto n^{1·3}$ nearly.

TABLE X.—March 23rd, 1892. ($l=7·025$ centim.)

$\theta°$ C.	n.	D.	y.
82	17·5	16	2·06
82	14	11	1·23
82	26	26	3·35
81	32	33	4·22
80	11·5	8·5	1·08
82	10	6·5	0·84

If it is assumed that $n=10$ is not much above the critical speed, μ may be calculated as 0·0646. According to (9) $\mu=0\cdot062$ for 81° C.

$y \propto n^{1\cdot30}$ may be taken as the law.

TABLE XI.—March 24th, 1892. ($l=6\cdot7$ centim.)

θ° C.	n.	D.	y.	μ.
65	9	8	1·07	·091
66	10·8	9	1·21	·086
65·5	17	16	2·14	·097
64·5	21	21	2·79	
65	23·5	26	3·46	
65	29	32	4·26	
65	114	210	28·00	
66	102	162	21·82	
64·5	88	156	20·10	
65	66	112	14·93	
66	52	74	9·96	
66	44·5	59	7·94	
66	38	46	6·19	

y is $D/(l+\lambda)$ corrected to 65° C. μ has no meaning except for the first three speeds, and the mean of these three is 0·091. According to (9) the value of μ for 65° C. is 0·085.

Above the critical speed, which is possibly below $n=17$, the law is probably $y \propto n^{1\cdot32}$.

It is not worth while to publish any of the observations which we have made upon other liquids, nor to publish the curves we have drawn for sperm-oil, although we exhibit them before the Society. Errors of one degree in observing temperature were quite possible, and errors of half a degree in the deflexion of our pointer were also possible. Small fluctuations in speed were continually taking place, so that the pointer was never quite still, the motion of the fluid was therefore not truly steady. It is our determination to repeat the whole work with improved apparatus. In the meantime, however, it will be observed from Table III. that there is fair agreement in the law connecting μ with temperature, from all the sets of observations. There is, on the whole, a very fair agreement with what we venture to call

Prof. Reynolds's rule,

$$y = a\mathrm{F}^{2-\kappa} n^{\kappa},$$

where κ has the value 1·33 or 1 according as n is above or below the critical speed*. The sheet of squared paper on which we have plotted all our values of $\log y$ and $\log n$ for the various constant temperatures shows that the errors of observation are too great for the establishment of this value of κ; but it is the probable value. It shows, however, in the allineation of the points of discontinuity, with sufficient accuracy that $y_c \propto n_c^2$, if the rule is taken to be generally true; and although there is some little vagueness always in one's observations just about the critical speed, we may take $y_c = 0{\cdot}009\, n_c^2$ without very great error. Indeed, we are satisfied with the substantial agreement of all our observations with the formula

$$y = a\left(\frac{\mu}{\cdot 769\, a}\right)^{2-\kappa} n^{\kappa},$$

where $a = {\cdot}009$ and $\kappa = 1{\cdot}33$. That is, for low speeds we have the law

$$y = \frac{\mu}{\cdot 769}\, n.$$

At the critical speed the law suddenly changes to

$$y = a\left(\frac{\mu}{\cdot 769\, a}\right)^{\frac{2}{3}} n^{1\frac{1}{3}},$$

which holds for all higher speeds which we have tried. The critical speed

$$n_c = \frac{\mu}{\cdot 769\, a} = 144\, \mu,$$

and

$$y_c = a n_c^2, \text{ or } {\cdot}009\, n_c^2.$$

It is to be recollected that ρ is too nearly constant for us to say with certainty that a is proportional to ρ, as the theory requires. The errors of observation were so great that it was not worth while finding accurately the most probable values of κ and a.

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* Prof. Reynolds, in criticizing a proof of this paper, has been kind enough to point out that his rule for pipes does not necessarily apply to the fluid in our apparatus. We had not seen the reprint of his Royal Institution lecture, else we should have known that the condition of the liquid in circular flow is inherently stable or unstable according as r is greater or less than the radius of the fixed cylindric surface. As he points out, the liquid in the outer space is inherently stable for velocities

We wish it to be understood that our apparatus was very carefully constructed, and great care was taken in making the observations; but it is our intention to pursue the investigation with apparatus much more carefully constructed.

Vibratory Experiments.

In designing the apparatus it was our intention to obtain μ from the damping of the rotational oscillations of the suspended cylinder about its vertical axis, the trough being at rest. We meant in this way to obtain μ for velocities very much smaller than those which could be employed in our steady motion experiments. A considerable number of observations were made, but when we tried to make calculations of μ we found that our mathematical difficulties were too great, and after many months of effort we are forced to say that we are unable to utilize these observations. In equation (4) assume that $v=w\epsilon^{ikt}$, and w may be obtained in Bessel functions. Unfortunately, as there are two surface conditions, *both* particular solutions of the Bessel equation are necessary, and the work of reduction becomes very great. An approximate solution is obtained by taking $r=\mathrm{R}+x$, R being the radius of the suspended cylinder, and taking the equation (4) to be

$$\frac{d^2v}{dx^2}+\frac{1}{\mathrm{R}}\frac{dv}{dx}-\frac{v}{\mathrm{R}^2}=\frac{\rho}{\mu}\frac{dv}{dt}. \quad . \quad . \quad . \qquad (11)$$

Making this assumption in the case of steady motion, it was found that it was sufficiently correct for practical purposes. The following numbers show the sort of error introduced, taking $\mathrm{R}=10$, and 1 the greatest value of x.

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far exceeding the critical velocity (if there is one) for plane surfaces, whereas the liquid in the inner space is unstable from the first.

We directed the attention of the meeting to the fact that Tables IV., V., VI., and VIII. give unmistakable evidence of the truth of what we have called Prof. Reynolds's Rule, however difficult we may find it in explanation.

Values of x.	Values of v.		
	Correct.	Approximate.	On the assumption of motion in plane layers.
0	1	1	1
·2	·7914	·7914	·8
·5	·4875	·4872	·5
·7	·2895	·2892	·3
1·0	0	0	0

The solution of (11) for vibratory motion is easy enough; but we found it still difficult to calculate μ from our observations. Even when we assume that the motion is in plane layers, so that the solution used by Maxwell is employed, we find that our $\frac{\mu}{\rho}$ is too great for a logarithmic decrement to exist with such amplitudes and times of oscillation as we had employed in the experiments, and it was impossible for us to repeat the experiments under the same conditions again at slower velocities, because the apparatus had been taken to pieces and could not be fitted up again in exactly the same way. When we say that a logarithmic decrement did not exist, we mean that it was not constant, but varied with the amount of the oscillation. For the tractive force to be proportional to the velocity of the cylinder it is necessary for μ/ρ and the periodic time to be so great that the velocities of the fluid at all places shall be in the same proportion as if the motion were steady.

After this paper was written we asked Mr. J. B. Knight, of the Chemical Department of the Finsbury Technical College, to make measurements of the specific gravity of sperm-oil at different temperatures. His results give a very striking confirmation of the views expressed in the paper as to a discontinuity of some kind due to rise of temperature. As all the authorities whom we have consulted seemed to see no possible reason for a discontinuity in the rate of change of μ with temperature in sperm-oil at about 40° C., it is possible that these results may be of importance.

Temperature Cent.	Specific Gravity.
25°	·831
30	·8306
35	·828
40	·826
45	·8758
50	·8753
55	·8717

We are now arranging a piece of apparatus which will give, not the absolute value of the specific gravity, but with great accuracy relative rates of the change of specific gravity with temperature *. We shall make experiments of the same kind upon other animal oils.

DISCUSSION.

After concluding the paper, Prof. Perry read a letter he had received from Prof. Osborne Reynolds on the subject, who doubted whether the true critical velocities had been reached in the experiments. In the particular arrangement employed, he would expect no critical velocity in the outer ring of liquid, whilst in the inner ring the motion would be unstable from the first.

Prof. Lodge said he had tried whether the refraction or the density of sperm-oil exhibited a discontinuity about 40°, but found none.

Mr. Rogers pointed out that experiments which corroborated those of Prof. Perry had been made by M. Couette, and published in *Ann. de Chim. et de Phys.* [6] xxi.

Mr. E. W. Smith suggested that the apparent discontinuity might be due to the separation of the constituents of an impure oil at certain temperatures and speeds of revolution.

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* We described at the Meeting results obtained for other specimens of sperm-oil, with the new apparatus, which exhibited no discontinuity. Yet we can find no reason to doubt Mr. Knight's measurements.

XVI. *On the Applicability of* Lagrange's *Equations of Motion in a General Class of Problems; with especial reference to the Motion of a Perforated Solid in a Liquid. By* Charles V. Burton, *D.Sc.**

1. Let ψ, ϕ, . . . be *some only* of the coordinates of a material system, so that when the values of ψ, ϕ, . . . are given the whole configuration is *not* completely determinate. But suppose it known that the kinetic energy T can be expressed *as a homogeneous quadratic function of* $\dot{\psi}$, $\dot{\phi}$, . . . *only*; so that we may write

$$\left.\begin{array}{l} 2\mathrm{T}=(\psi\psi)\dot{\psi}^2+2(\psi\phi)\dot{\psi}\dot{\phi}+\ldots \\ (\psi\psi),\ (\psi\phi),\ldots \text{ are functions of } \psi,\ \phi,\ldots \text{ only} \end{array}\right\}\quad . \quad . \quad (1)$$

We also suppose it known *that* (1) *continues to hold good so long as the only (generalized) forces and impulses acting are of types corresponding to*

$$\psi,\ \phi, \ldots . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

2. Suppose, now, that such impulses of these types were to act on the system that $\dot{\psi}$, $\dot{\phi}$, . . . were all reduced to zero; the expression for the kinetic energy would accordingly vanish, and the system would be at rest. By supposing the last operation to be reversed, we see that *the motion at any instant could be produced from rest by impulses of the types corresponding to*

$$\psi,\ \phi, \ldots \textit{only}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

3. Let x, y, z be the Cartesian coordinates at time τ of a mass-element m referred to fixed axes, and let T be the kinetic energy of the system at the same instant. Further, let A be the "action" when the system moves without additional constraint from one configuration to another, and A + δA the action when by workless constraints the path is slightly

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* Read March 10, 1893.

modified, so that in place of the coordinates x, y, z we have $x+\delta x, y+\delta y, z+\delta z$. Then*

$$\delta\mathrm{A}=\{\Sigma m(\dot{x}\delta x+\dot{y}\delta y+\dot{z}\delta z)\}-[\Sigma m(\dot{x}\delta x+\dot{y}\delta y+\dot{z}\delta z)]$$
$$+ \text{ a term which necessarily vanishes}; \quad . \quad . \quad (4)$$

where [ ] and { } denote the values of the quantities enclosed at the beginning and end of the motion considered.

Suppose further that, both at the beginning and at the end, the values of $\psi, \phi, \ldots$ are the same for the one motion as for the other, so that initially and finally $\delta\psi, \delta\phi, \ldots$ are all zero. *It does not follow that all the $\delta x, \delta y, \delta z$'s are zero*; but

$$\Sigma m(\dot{x}\delta x+\dot{y}\delta y+\dot{z}\delta z)$$

is the so-called "virtual moment" of the actual momenta in the hypothetical displacement $\delta x, \delta y, \delta z$; that is, the virtual moment, in the same displacement, of the impulse necessary to produce the actual motion from rest. In virtue of (3), therefore, and of the initial and final vanishing of $\delta\psi, \delta\phi, \ldots$ we see that the bracketed terms of (4) must both be zero; hence

> *The increment $\delta\mathrm{A}$ vanishes and A has a stationary value for all worklessly effected variations of path which leave the initial and final values of $\psi, \phi, \ldots$ unaltered.* . (5)

4. Lagrange's equations for the coordinates $\psi, \phi, \ldots$ may now be written down at once, since the investigation of Thomson and Tait† becomes applicable to the present case without modification. It will be noticed that in their equations $(10)^{\mathrm{v}}$ and $(10)^{\mathrm{vi}}$, § 327, the sign of $\partial\mathrm{V}/\partial\psi$ should be reversed.

We have thus a perfectly general proof of the proposition: *If the kinetic energy of a material system can be expressed as a homogeneous quadratic function of certain generalized velocities $\dot{\psi}, \dot{\phi}, \ldots$ only, the coefficients being functions of $\psi, \phi, \ldots$ only, and if this remains always true so long as the only forces and*

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* Thomson and Tait's 'Natural Philosophy,' 2nd edit. Part I. § 327.

† *Loc. cit.*

impulses acting are of types corresponding to ψ, ϕ, . . . , the equations of motion for the coordinates ψ, ϕ, . . . may be written down from this expression for the energy, in accordance with the Lagrangian rule. Provided only that the stated conditions are satisfied, we need not consider whether the whole configuration is determined by the values of ψ, ϕ, . . . , or what is the nature of the ignored coordinates. . . . . (A)

5. Passing over the known application of this result to the motion of solids through an irrotationally and acyclically moving liquid, we come to the more general case of a perforated solid, with liquid irrotationally circulating through the apertures. Take as coordinates any six θ, θ', . . . which determine the position of the solid, together with χ, χ', . . . equal in number (m) to the apertures; each χ being the volume of liquid which, starting from a given configuration, has flowed across some one of the m geometrical surfaces, required to close the apertures, these surfaces being supposed to move along with the solid.

Of course the coordinates θ, θ', . . . χ, χ', . . . are insufficient to determine the entire configuration of the system (including the positions of all the particles of liquid); but we shall see immediately how, in virtue of the proposition (A), Lagrange's equations may be written down.

6. Since an increment $\delta\chi$ in one of the coordinates χ is the volume of liquid which flows across a barrier-surface (*i. e.*, which flows through an aperture *relatively to the solid*), the generalized force corresponding to χ must be conceived of as a uniform pressure exerted over the said geometrical surface, by means of some immaterial mechanism attached to the solid; while the impulse corresponding to χ is of course a uniform impulsive pressure applied in the same manner. From hydrodynamical considerations we know that the measure of such an impulsive pressure is $\rho\delta\kappa$, where ρ is the density of the fluid, and $\delta\kappa$ the change produced in the circulation through the corresponding aperture.

Hence the impulses corresponding to χ, χ', . . . are

$$\kappa\rho,\ \kappa'\rho,\ \ldots\ldots\ldots \qquad (6)$$

where κ, κ',... are the circulations through the various apertures.

7. Now when the motion of the liquid is irrotational, we have

T = a homogeneous quadratic function of $\dot\theta, \dot\theta', \ldots \kappa, \kappa' \ldots$ only; coefficients functions of $\theta, \theta' \ldots$ only;

$\dot\chi, \dot\chi', \ldots$ = homogeneous linear functions of $\dot\theta, \dot\theta', \ldots \kappa, \kappa', \ldots$ only; coefficients functions of $\theta, \theta', \ldots$ only.

Since the $\dot\chi$'s are equal in number to the κ's, let us suppose the last-written system of linear equations to be solved for the κ's in terms of the $\dot\chi$'s; we then have

$\kappa, \kappa', \ldots$ = homogeneous linear functions of $\dot\theta, \dot\theta', \ldots \dot\chi, \dot\chi', \ldots$ only; coefficients functions of $\theta, \theta', \ldots$ only.

Substituting in the expression for T we get

T = a homogeneous quadratic function of $\dot\theta, \dot\theta', \ldots \dot\chi, \dot\chi', \ldots$ only; coefficients functions of $\theta, \theta', \ldots$ only.

This, then, remains true so long as the motion of the liquid is irrotational; in other words, so long as the only forces and impulses acting are of types corresponding to $\theta, \theta', \ldots$ (since these are applied to the solid), $\chi, \chi', \ldots$ (since these are uniform over the barriers, by § 6).

If we identify ψ, ϕ,... with the coordinates $\theta, \theta', \ldots \chi, \chi', \ldots$ of the present example, we see that the proposition (A) of § 4 is immediately applicable to this case. We may therefore ignore all other coordinates, and from the kinetic energy expressed as a function of $\dot\theta, \dot\theta', \ldots \dot\chi, \dot\chi', \ldots$ write down the Lagrangian equations for $\theta, \theta', \ldots$ and, if we wish, for $\chi, \chi', \ldots$ also. These latter, however, are less directly intelligible, since in general they involve finite pressures continuously acting over geometrical surfaces drawn through the liquid.

8. If we wish to picture the application of the principle of least action (§ 3) to the present case, we may proceed as follows:—Let the system start from the configuration (I.) and move without additional constraint or influence to the configuration (II.). Then let it start again from the configuration (1.) with the same velocities as before, and during the motion let infinitesimal additional forces act on the solid,

while infinitesimal pressures, uniform over each barrier-surface, are impressed on the liquid; the total rate at which the additional influences do work being at each instant zero. Further, let the additional influences be so adjusted that the system, after following a slightly different path, passes through a configuration such that $\theta, \theta', \ldots \chi, \chi', \ldots$ are all the same as for (II.). Then, to pass from the configuration (II.) to the present configuration requires no displacement of the solid, and only such displacement of the liquid that the *total* volume which crosses any barrier-surface is zero. In such a change of configuration impulses of the types $\theta, \theta', \ldots \chi, \chi', \ldots$ would have no "virtual moment," just as forces applied to the solid and uniform pressures applied to the barrier-surfaces would give rise to no virtual work.

9. At this stage it will be convenient to replace $\dot{\theta}, \dot{\theta}', \ldots$ by the components u, v, w of linear velocity and p, q, r of angular velocity, which determine the instantaneous motion of the solid along and about axes fixed in itself. The Lagrangian equations for the six coordinates $\theta, \theta', \ldots$ must accordingly be replaced by the forms suitable to moving axes. The expression for the energy in terms of the velocities now becomes a homogeneous quadratic function of $u, v, w, p, q, r, \dot{\chi}, \dot{\chi}', \ldots$ in which all the coefficients are known to be constants.

Let us apply the method due to Routh [*], and *modify* this function with respect to the coordinates $\chi, \chi', \ldots$ If T be the value of the kinetic energy in terms of the velocities alone, the modified function (*i.e.* the kinetic part of Routh's modified Lagrangian function)

$$\mathrm{T}' = \mathrm{T} - \frac{\partial \mathrm{T}}{\partial \dot{\chi}}\dot{\chi} - \frac{\partial \mathrm{T}}{\partial \dot{\chi}'}\dot{\chi}' - \ldots = \mathrm{T} - \kappa\rho\chi - \kappa'\rho\dot{\chi}' - \quad . \; . \; (7)$$

from (6). It is further known that the whole energy of the system

$$= \mathrm{E} + \mathrm{K}, \quad . \; . \; . \; . \; . \; . \; . \; . \; (8)$$

where E is a function of $u \ldots, p \ldots$, only, and K is a function of the momenta $\kappa\rho$ only. Suppose, now, that the solid were brought to rest by forces applied to it alone: E would vanish along with u, v, w, p, q, r, while the circulations κ, and consequently also K, would remain unaltered. The generalized velocities $\dot{\chi}$, $\dot{\chi}'$, .. would in general have changed, becoming, let us suppose $\dot{\chi}_0$, $\dot{\chi}_0'$,... and the kinetic energy would accordingly have become

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[*] 'Rigid Dynamics,' vol. i. chap. viii.

$$K=\tfrac{1}{2}(\kappa\rho\dot{\chi}_0+\kappa'\rho\dot{\chi}_0'+\ldots). \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Now let

$$\dot{\chi}=\dot{\chi}_0+\dot{\chi}_1,\quad \dot{\chi}'=\dot{\chi}_0'+\dot{\chi}_1', \quad . \quad . \quad . \quad . \quad . \quad (10)$$

so that each $\dot{\chi}_1$ is that part of the flux of liquid (volume per unit time) which takes place across a barrier-surface owing to the motion of the solid itself.

Having regard to (8), (9), and (10) our equation (7) for T' becomes

$$T'=(E+K)-2K-\kappa\rho\dot{\chi}_1-\kappa'\rho\dot{\chi}_1'. \quad . \quad . \quad . \quad (11)$$

Let us write for the velocity-potential of the *acyclic* motion

$$\Phi=u\phi_u+v\phi_v+w\phi_w+p\phi_p+q\phi_q+r\phi_r, \quad . \quad . \quad (12)$$

and for the value of $\dot{\chi}_1$ across the barrier-surface σ we have

$$\dot{\chi}_1=\iint\left\{\frac{\partial\Phi}{\partial\nu}-[ul+vm+wn+p(ny-mz)\right.$$
$$\left.+q(lz-nx)+r(mx-ly)]\right\}d\sigma, \quad . \quad . \quad . \quad (13)$$

where x, y, z are the coordinates of the element $d\sigma$ and l, m, n are the direction-cosines of its normal ν, all referred to the system of axes fixed in the solid. From (12) and (13) substitute in (11); thus

$$T'=E-K+u\Sigma\kappa\rho\iint\left(l-\frac{\partial\phi_u}{\partial\nu}\right)d\sigma+\text{similar terms in } v, w,$$

$$+p\Sigma\kappa\rho\iint\left(ny-mz-\frac{\partial\phi_p}{\partial\nu}\right)d\sigma+\text{similar terms in } q, r, \quad . \quad (14)$$

where the summation refers to the m barriers.

Remembering (8) it will be seen that T′ is now expressed in the proper form, namely as a function of u, v, w, p, q, r, and the momenta $\kappa\rho$, $\kappa'\rho, \ldots$ only. By means of the relations

$$\frac{d}{dt}\frac{\partial \mathrm{T}'}{\partial u} - r\frac{\partial \mathrm{T}'}{\partial v} + q\frac{\partial \mathrm{T}'}{\partial w} = \mathrm{X}, \text{ \&c., \&c.}$$

$$\frac{d}{dt}\frac{\partial \mathrm{T}'}{\partial p} - w\frac{\partial \mathrm{T}'}{\partial v} + v\frac{\partial \mathrm{T}'}{\partial w} - r\frac{\partial \mathrm{T}'}{\partial q} + q\frac{\partial \mathrm{T}'}{\partial r} = \mathrm{L}, \text{ \&c., \&c.}$$

the equations of motion of the solid can at once be written down. X, . . ., L, . . ., are of course impressed force- and couple-constituents.

10. Since the kinetic energy due to any number of perforated solids, moving in circulating liquid, can be divided into two parts, of which one is a function of the component velocities of the solids alone, and the other a function of the circulation-momenta alone, the above method may obviously be extended; in fact a slight change in (14) will render it at once applicable to the more general case. We shall have, evidently,

$$\mathrm{T}' = \mathrm{E} - \mathrm{K} + \Sigma u \Sigma \kappa\rho \iint \left(l - \frac{\partial \phi_u}{\partial \nu}\right) d\sigma + \text{similar terms in } v, w,$$

$$+ \Sigma p \Sigma \kappa\rho \iint \left(ny - mz - \frac{\partial \phi_p}{\partial \nu}\right) d\sigma + \text{similar terms in } q, r, \quad (15)$$

where E is still the energy due to the motion of the solids and the acyclic motion of the liquid, and K the energy due to the circulations. In each barrier-term the first Σ denotes summation with respect to all the solids, and for each u or p, &c., the second Σ denotes summation with respect to all the barriers of the system.

These hydrodynamical results are not new, but the method of proof is in some respects different from anything that has yet been given, and will, I hope, be found intelligible and fairly simple. In an admirable memoir, just communicated to the Physical Society, Mr. Bryan has given a direct hydrodynamical proof of the equations holding good for the motion of the system in question; but it seemed to me also desirable that the problem should be rigorously treated by the method of generalized coordinates, avoiding any assumption as to the impulse of the cyclic motion, and proceeding entirely from the principles established by Lagrange, and extended by Hamilton, Routh, and Hayward.

When this paper was in proof it contained some remarks on the ignoration of coordinates, as treated in Thomson and Tait's 'Natural Philosophy'*. Calling $\chi, \chi', \ldots$ the independent coordinates which, together with $\psi, \phi, \ldots$ determine the whole configuration of the system in §§ 1,..., it was suggested that, in hydrodynamical and kindred applications, there was a difficulty in proving that $\partial T/\partial\chi$, $\partial T/\partial\chi', \ldots$ were all zero.

But the difficulty, if indeed it should exist, is easily removed. For since the actual motion at any instant could be generated from rest by impulses of types corresponding to $\psi, \phi, \ldots$ only, we have throughout the motion

$$\partial T/\partial\dot{\chi}=0, \quad \partial T/\partial\dot{\chi}'=0, \ldots;$$

and by the Lagrangian equations for $\chi, \chi', \ldots$, since all the generalized forces are of types corresponding to $\psi, \phi, \ldots$, we get

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\chi}}-\frac{\partial T}{\partial\chi}=0, \quad \frac{d}{dt}\frac{\partial T}{\partial\dot{\chi}'}-\frac{\partial T}{\partial\chi'}=0, \ldots$$

whence

$$\frac{\partial T}{\partial\chi}=0, \quad \frac{\partial T}{\partial\chi'}=0, \ldots.$$

DISCUSSION.

A criticism by Mr. A. B. Basset on Mr. Bryan's recent paper, and also on Dr. Burton's paper, was read by Mr. Elder.

Mr. Basset regards the process employed by Mr. Bryan in obtaining the equations of motion as a distinctly retrograde step, and thinks the most scientific way of dealing with dynamical problems is to avoid the unnecessary introduction of any unknown reactions. The advantages of the theory of the impulse are described by Mr. Basset, and the parts which require care when applying the theory to cyclic irrotational motion pointed out. Comparisons are then made as regards simplicity, between the different methods of treating the subject which have been used by Mr. Bryan, Prof. Lamb, and himself.

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* Part I. § 319, example G.

With reference to Dr. Burton's paper, he thinks it will tend to complicate rather than elucidate the subject.

An account of how Lagrange's original equations had been modified by Hamilton, Routh, and himself is given at some length, and the advantages and power of the mixed transformation which he had developed are pointed out.

Prof. Henrici said he agreed with Mr. Basset, in preferring the more general method, but thought the independent treatment of special problems, as given by Mr. Bryan and Dr. Burton, very desirable.

Dr. Burton, in reply, said he concurred with Mr. Basset on some points, but thought it decidedly advantageous to look at problems from different points of view. The investigation he (Dr. Burton) had given was applicable to any number of solids, and, on the whole, simpler than Mr. Basset's.

The President pointed out that no attack had been made on the validity or accuracy of Mr. Bryan's or Dr. Burton's work. As to simplicity of the various methods, different opinions might be expected to exist. He himself thought it very desirable that such problems should be approached from different sides.

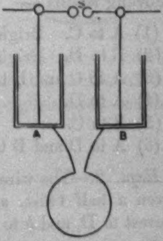
XVII. *Luminous Discharges in Electrodeless Vacuum-Tubes.*

By E. C. RIMINGTON*.

SINCE reading a paper in conjunction with Mr. E. W. Smith on November 25th, 1892, before this Society†, on "Experiments in Electric and Magnetic Fields, Constant and Varying," the Author's attention has been drawn to a paper contributed by Mr. Tesla to the 'Electrical Engineer' of New York, July 1st, 1891, in which the luminous ring-shaped discharge obtained when a Leyden jar is discharged through a coil of wire surrounding an exhausted bulb is attributed to the electrostatic action of the surrounding wire, and not to the electric stress set up in the rarefied dielectric in consequence of the rapidly oscillating magnetic induction through the bulb.

As one experimental proof of this assertion Mr. Tesla gives the following experiment :—"An ordinary lamp-bulb was surrounded by one or two turns of thick copper wire, and a luminous circle excited by discharging the jar through this primary. The lamp-bulb was provided with a tinfoil coating on the side opposite to the primary, and each time the tinfoil coating was connected to the ground, or to a large object, the luminosity of the circle was considerably increased."

Fig. 1.



The author repeated this experiment with two Leyden jars arranged as in fig. 1, and found that when the spark-gap was sufficiently large to produce a bright ring when the tinfoil was not connected to earth, doing so produced no noticeable difference in the brilliancy ; but that, if the discharge were faint, it was renrendered considerably brighter on making the earth connexion. Better results were, however, obtained on connecting the tinfoil to either of the outside coatings, A or B, of the jars instead of to earth. This result led the author to try a series of experiments to endeavour to determine the cause of the effect, of which the typical ones are here given.

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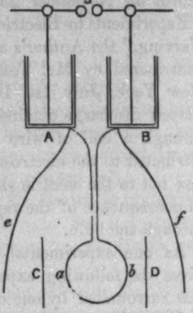
* Read April 28, 1893.

† *Ante*, p. 3.

Fig. 2.



Experiment 1 (*vide* fig. 2). A and B are the outside coatings of a pair of Leyden jars (those employed were about pint size). C and D two vertical and parallel metal plates, at a distance of about one foot from the jars. The spark-gap, S, is adjusted by a screw, so that the spark-length can be varied by small amounts when necessary. A single turn of wire, *a b*, encloses an exhausted bulb, and its ends are connected to A and B, as shown in the figure, so that *a* the part nearest to C is connected to A, and *b* to B. Two loose wires, *e* and *f*, are also connected to A and B.

The spark-gap is now shortened until there is just no luminous ring in the bulb.

The plates C and D are then connected to the outer coatings A and B by means of the two loose wires, with the following results :—

(1) A to C. Bright ring.
(2) B to D. Bright ring.
(3) A to C and B to D simultaneously. Bright ring.
(4) A to D. }
(5) B to C. } No luminous ring.
(6) A to D and B to C. }

Expt. 2.—The wire turn *a b* is removed from the bulb, given a half twist, and then replaced ; so that *a* is now nearest to D, and *b* to C. Plates not connected, no luminous ring.

(1) A to C. }
(2) B to D. } No luminous ring.
(3) A to C and B to D. }

(4) A to D.
(5) B to C.
(6) A to D and B to C.
} Bright ring.

Expt. 3.—Arranged as in Expt. 1, case (1) or (2). C is then connected to D, and the ring becomes less bright.

Expt. 4.—Arranged as in Expt. 1, case (1). C and D connected. On approaching C to the bulb, ring becomes brighter.

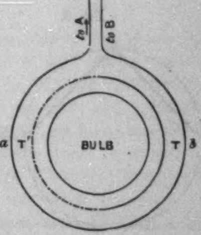
On approaching D less bright.

If arranged as Expt. 1, case (2), the reverse happens.

All the above four experiments give the same effects if the turn of wire be larger than the bulb, as in fig. 3, only a longer spark-gap has to be used.

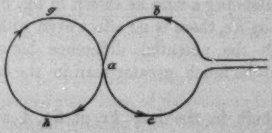
Expt. 5.—A single turn of wire (fig. 3), *a b*, larger than the bulb is employed, and between the bulb and the ring a semicircular strip of tinfoil or metal T is placed. The wire is connected as in Expt. 1. The spark-gap is arranged to give no ring. Connecting T to B bright ring, T to A no ring. The reverse happens if the tinfoil is placed in position T' as shown by the dotted line.

Fig. 8.



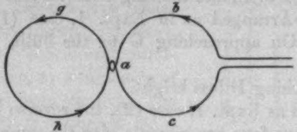
Expt. 6.—A piece of gutta-percha covered wire is bent into shapes shown in figs. 4 and 5. On placing either of these over bulb as in fig. 6, a figure-of-eight-shaped luminous discharge is obtained, and there is no noticeable difference between the two.

Fig. 4.



Expt. 7.—Putting the wire (fig. 4) on bulb as in fig. 7, a single broad band-ring is obtained, as the two turns will help one another with respect to magnetizing effect.

Fig. 5.



Doing the same with the wire (fig. 5) a discharge is obtained shaped like the sector of an orange, as shown by the dotted lines, fig. 7.

Fig. 6.

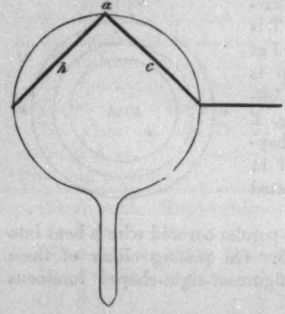
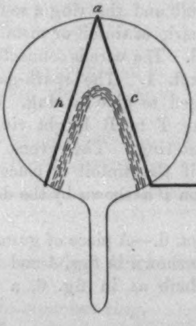


Fig. 7.

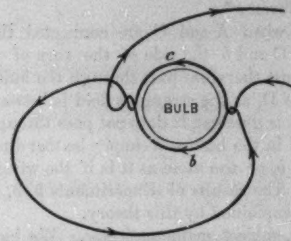


Expt. 8.—Bending a wire as shown in fig. 8, and placing a bulb in the loop *b c*, there is no effect even with a long spark-gap, although the potential difference between the sides *c* and *b* would be much greater than in the case of a single turn.

Putting a bulb in the loop, *b c*, of fig. 4 at once gives a bright ring.

Experiments 6, 7, and 8 seem to show that ring, or other shaped, sharp luminous discharges can only be obtained with the wire so wound as to give magnetic induction through the bulb, while the first five experiments show that an electrostatic field in the bulb may help the effect. The theory the author has come to after consideration of the above and other experiments is :—That if the E.M.F. due to rate of change of magnetic induction acting in the dielectric of rarefied gas be insufficient to break it down and produce a luminous discharge (owing to the spark-gap being too short), the electrostatic field between the plates C and D, or between one of the plates and part of the wire, if correctly timed with respect to the rate of change of current in the wire, will commence the breakdown of the gas, thus allowing a less E.M.F. due to the magnetic induction to complete it.

Fig. 8.



To put this to the test, a single turn of wire was put round a bulb and the spark-gap adjusted so as to give a very faint or no luminous ring ; on the top of the bulb was laid a piece of tinfoil connected to one pole of a ¼ in. spark induction-coil ; when the coil is worked the tube is filled with a faint glow : if now the Leyden jars are charged and discharged there will be sometimes a ring in the bulb which will be occasionally quite bright. The reason it cannot be always bright is of course that the discharges of the induction-coil are periodic, as are also those of the jars, and it is only when the two are properly timed (*i.e.* the P.D. due to the coil coming either just before or simultaneously with the spark) that there will be a bright ring.

This experiment seems to settle the question and show conclusively that a properly timed electric stress in the bulb due to an electrostatic field will allow an E.M.F. due to the alternating current in the wire to produce a breakdown of the rarefied gas, which the latter is too small to effect without the aid of the former*.

In Expt. 1, when A and C are connected this field will exist between C and *b* the side of the turn of wire remote from C, and must therefore pass through the bulb. When A is connected to D, as the strongest field is between *b* and D, where the P.D. is greatest, it does not pass through the bulb; in fact the field in the bulb will simply be that due to the P.D. between *a* and *b*, or the same as it is if the wires *e* and *f* are disconnected. The results of Experiments 2, 3, 4, and 5 are also obviously explained by this theory.

To treat the subject mathematically. We have the well-known equations for the discharge of a condenser:

$$L\frac{dc}{dt} + Rc = -\frac{q}{K}, \text{ where K is the capacity,}$$

and

$$c = \frac{dq}{dt}.$$

Combining these,

$$\frac{d^2q}{dt^2} + \frac{Rdq}{Ldt} + \frac{1}{KL}q = 0.$$

To obtain an oscillatory discharge 4L must be greater than KR^2.

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* To prevent misconception, it had better be definitely stated that this electrostatic stress does not necessarily act in the same direction as the E.M.F. due to the rate of change of magnetic induction. In experiments (1) to (5) the direction of the former will be through the bulb from side to side, while that of the latter is a circle coplanar with the wire. As the discharge in a gas is of a nature more or less electrolytic, being accompanied by the splitting up of the molecules, it seems reasonable to suppose that anything which increases the number of dissociated molecules will enable a smaller stress to produce a breakdown in the form of a luminous discharge.

Putting a for $-\frac{R}{2L}$ and b for $\sqrt{\frac{1}{KL}-\frac{R^2}{4L^2}}$, the solution is

$$q=Qe^{at}\ \frac{\sqrt{a^2+b^2}}{b}\sin(bt+\theta), \quad . \quad . \quad . \quad (1)$$

where $\theta=\tan^{-1}\left(-\frac{b}{a}\right)$ and Q is the initial charge.

This may be more conveniently written

$$q=Qe^{at}\frac{\sqrt{a^2+b^2}}{b}\cos(bt-\eta),$$

where

$$\eta=\frac{\pi}{2}-\theta, \text{ or } \tan\eta=-\frac{a}{b}=\sqrt{\frac{KR^2}{4L-KR^2}}.$$

If the oscillations are to be rapid, $\frac{1}{KL}$ must be large compared to $\frac{R^2}{4L^2}$.

Therefore η will be some small angle.

Instead of quantity we may write P.D. of the condenser, or

$$v=Ve^{at}\ \frac{\sqrt{a^2+b^2}}{b}\cos(bt-\eta). \quad . \quad . \quad . \quad (2)$$

The current

$$c=\frac{dq}{dt}=\frac{Q}{bKL}e^{at}\sin bt=\frac{V}{bL}e^{at}\sin bt. \quad . \quad . \quad . \quad (3)$$

Now the electric stress acting in the bulb is proportional to the rate of change of current, or to $\frac{dc}{dt}$;

and $$\frac{dc}{dt}=\frac{V}{bL}e^{at}(a\sin bt+b\cos bt). \quad . \quad . \quad . \quad . \quad (4)$$

The current itself will be a maximum or minimum when

$$\frac{dc}{dt}=0\,;$$

i.e. when $$a\sin bt+b\cos bt=0,$$

or when $$\tan bt=-\frac{b}{a}=\sqrt{\frac{4L}{KR^2}-1}.$$

Therefore $bt=\theta$, and is in general nearly equal to $\frac{\pi}{2}$.

The maximum values of the current occur when

$$bt=\theta,\quad 2\pi+\theta,\quad 4\pi+\theta,\ \&c.,$$

and the minimum values when

$$bt=\pi+\theta,\quad 3\pi+\theta,\quad 5\pi+\theta,\ \&c.$$

This is shown in the curve (fig. 9), the points M_1, M_2, M_3, &c., representing the maximum and minimum values of the current. The distance O A represents θ, and A B$=\frac{\pi}{2}-\theta=\eta$.

Fig. 9.

Dotted curves are values of the exponential $\frac{V}{bL}e^{-\frac{R}{2L}t}$.

M_1, M_2, M_3, M_4 are the maximum values of the current. μ_1, μ_2, μ_3 are the points where the rate of change of current is greatest.

It is now necessary to consider when the rate of change of the current is greatest. $\frac{dc}{dt}$ will be a maximum or minimum when $\frac{d^2c}{dt^2}=0$.

Now

$$\frac{d^2c}{dt^2}=\frac{\mathrm{V}}{b\mathrm{L}}e^{at}\{(a^2-b^2)\sin bt+2ab\cos bt\}=0.$$

Hence

$$\tan bt=-\frac{2ab}{a^2-b^2}=-\frac{\mathrm{R}\sqrt{\mathrm{K}(4\mathrm{L}-\mathrm{KR}^2)}}{2\mathrm{L}-\mathrm{KR}^2}.$$

Let

$$\tan^{-1}\frac{\mathrm{R}\sqrt{\mathrm{K}(4\mathrm{L}-\mathrm{KR}^2)}}{2\mathrm{L}-\mathrm{KR}^2}=\gamma.$$

γ will be in general a small angle.

The rate of change of current will be greatest (either a maximum or a minimum) when

$$bt=-\gamma,\ \ \pi-\gamma,\ \ 2\pi-\gamma,\ \&c.$$

Obviously bt cannot equal $-\gamma$, so that the rate of change of current is greatest for values $\pi-\gamma$, $2\pi-\gamma$, &c.; or at points μ_1, μ_2, μ_3, &c. in the curve (fig. 9), and $\mathrm{EF}=\gamma$. If the oscillations are to be very rapid KR^2 must be negligible compared to $4\mathrm{L}$; in which case

$$\tan\gamma=\mathrm{R}\sqrt{\frac{\mathrm{K}}{\mathrm{L}}};$$

also

$$\tan\eta=\frac{\mathrm{R}}{2}\sqrt{\frac{\mathrm{K}}{\mathrm{L}}},$$

and they are both very small angles, hence $\gamma=2\eta$ approximately, or $\mathrm{EF}=2\mathrm{AB}$.

When $bt=\pi-\gamma$,

$$\frac{dc}{dt}=\frac{\mathrm{V}}{\mathrm{L}}e^{-\frac{\pi-\gamma}{\sqrt{\frac{4\mathrm{L}}{\mathrm{KR}^2}-1}}},$$

or if the oscillations are very rapid,

$$\frac{dc}{dt}=\frac{\mathrm{V}}{\mathrm{L}}e^{-\frac{\pi\mathrm{R}}{2}\sqrt{\frac{\mathrm{K}}{\mathrm{L}}}}.$$

If, however, $t=0$,

$$\frac{dc}{dt}=\frac{\mathrm{V}}{\mathrm{L}},$$

so that the greatest rate of change of current occurs at the first instant of discharge, although this is not a mathematical maximum.

Equation (4) may also be written

$$\frac{dc}{dt}=\frac{\mathrm{V}\sqrt{a^2+b^2}}{b\mathrm{L}}e^{at}\cos(bt+\eta), \quad . \quad . \quad . \quad (5)$$

η being the same angle as before.

It is now necessary to consider the values of the P.D. between the outside coatings A and B of the Leyden jars.

Let l and r be the inductance and resistance of the coil connected to the outer coatings, and L and R the same for the whole circuit. Let $v_1-v_2=x$ be the P.D. between the outer coatings at any instant t. Then

$$c=\frac{\mathrm{V}}{b\mathrm{L}}e^{at}\sin bt,$$

and

$$x=v_1-v_2=cr+l\frac{dc}{dt},$$

$$\therefore \quad x=\frac{\mathrm{V}}{b\mathrm{L}}e^{at}\ \{(r+la)\sin bt+lb\cos bt\}$$

$$=\frac{\mathrm{V}}{b\mathrm{L}}e^{at}\ \sqrt{l^2b^2+(r+la)^2}\cos(bt-\eta') \quad . \quad . \quad (6)$$

where

$$\tan\eta'=\frac{\frac{r}{l}+a}{b}.$$

η' will be in general a small angle not very different from η; and if $\frac{l}{r}=\frac{\mathrm{L}}{\mathrm{R}}$ or the time-constant of the coil equals the time-constant of the whole circuit,

$$\frac{r}{l}=\frac{\mathrm{R}}{\mathrm{L}}=-2a,$$

$$\therefore \ \tan\eta'=-\frac{a}{b}=\tan\eta, \text{ or } \eta'=\eta.$$

That is x is in phase with v the P.D. at the inner coatings of the jars.

To find the maxima and minima of x we have

$$\frac{dx}{dt}=\frac{\mathrm{V}}{b\mathrm{L}}e^{at}\left[\{ar+l(a^2-b^2)\}\sin bt+b(r+2la)\cos bt\right]=0.$$

$$\therefore\ \tan bt=-\frac{b(r+2la)}{ar+l(a^2-b^2)}$$

$$=\frac{(r\mathrm{L}-\mathrm{R}\quad\sqrt{\mathrm{K}(4\mathrm{L}-\mathrm{KR}^2)}}{\mathrm{L}(\mathrm{KR}r+2\mathrm{L})}$$

$$=\tan\delta.$$

Then x has its greatest positive or negative values when $bt=\delta,\ \pi+\delta,\ 2\pi+\delta$, &c. δ is in general a small angle, and is positive if

$$\frac{\mathrm{L}}{\mathrm{R}}>\frac{l}{r},$$

and negative if

$$\frac{\mathrm{L}}{\mathrm{R}}<\frac{l}{r}.$$

If

$$\frac{\mathrm{L}}{\mathrm{R}}=\frac{l}{r};\quad \delta=0.$$

If $\frac{\mathrm{L}}{\mathrm{R}}$ be not greater than $\frac{l}{r}$ the first largest value of x will occur at $t=0$, and as the rate of change of current is also greatest at this instant the two will occur simultaneously.

The next greatest value of x occurs when

$$bt=\pi-\delta\ \left(\text{if}\,\frac{\mathrm{L}}{\mathrm{R}}<\frac{l}{r}\right),$$

and the next greatest rate of change of current when

$$bt=\pi-\gamma.$$

δ will be less than γ, if $\frac{\mathrm{L}}{\mathrm{R}}$ be nearly equal to $\frac{l}{r}$; so that the maximum value of x will occur after the maximum of $\frac{dc}{dt}$, but the value of x will not differ very much from its maximum when $\frac{dc}{dt}$ is a maximum *. This bears out the results obtained in experiments 1 and 2, though, of course, the electric field in the bulb will be that due to the P.D. between one of the plates, C or D, and the opposite side of the turn of wire, and this will only be about half that between the outer coatings A and B. Moreover, the phase of the potential of C will not be quite the same as that of A, on account of the inductance of the connecting wire *e*. Experiments 1 and 2 were, however, tried with the plates C and D, and the connecting wires removed, the turn of wire *ab* being moved so as to bring either *a* or *b* nearest to A or to B, and the results obtained were practically the same as those of experiments 1 and 2.

Effect of Size of Jars.

When different-sized Leyden jars are employed with the same length of spark-gap the luminous ring is more brilliant

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* The above investigation into the value of the P.D. between the outer coatings will only give correctly the state of things when a steady swing has been set up in the circuit; as evidently when $t=0$ the value of x also equals zero, so that x must start in phase with the current; it will, however, rapidly get out of phase with the latter, and finally be nearly in quadrature with it. This is due to an initial wave starting from the spark-gap which runs round the circuit. Possibly the value of can be empirically represented by one of the two subjoined formulæ :—

$$x=\frac{\mathrm{V}\epsilon^{at}}{b\mathrm{L}}\sqrt{l^2b^2+(r+la)^2}\sin\left\{(bt+\psi)(1-\epsilon^{-pt})\right\},$$

or

$$x=\frac{\mathrm{V}\epsilon^{at}}{b\mathrm{L}}\sqrt{l^2b^2+(r+la)^2}\sin\left\{bt+\psi(1-\epsilon^{-pt})\right\},$$

where $\psi=\frac{\pi}{2}-\eta'$, and p some constant. Dr. Lodge, in his researches on the A and B sparks, approximately represents the initial values of x by the current multiplied by the impedance of the conductor r, or makes

$$x=\frac{\mathrm{V}\epsilon^{at}}{b\mathrm{L}}\sqrt{l^2b^2+r^2}\sin bt.$$

The initial maximum of x will consequently roughly coincide with the maximum of the current, or be near the point M_1 of fig. 9, and will thus come about a quarter of a period before the second maximum rate of change of current, point μ_1 (fig. 9).

with larger jars. Now the E.M.F, acting in the rarefied gas, and producing the breakdown of the same, is proportional to $\frac{dc}{dt}$.

Also the greatest value of $\frac{dc}{dt}$ that first occurs is when $t=0$, and then

$$\frac{dc}{dt}=\frac{V}{L},$$

and the next is for very rapid oscillations

$$\frac{dc}{dt}=\frac{V}{L}e^{-\frac{\pi R}{2}}\sqrt{\frac{K}{L}}.$$

So that the first value of the E.M.F. acting in the gas is independent of the capacity, and the next and succeeding values are less the greater the capacity.

The effect on the eye, however, of the luminous ring will be the time-integral of the discharge or approximately depend on

$$\int_0^\infty \frac{dc}{dt}\,dt.$$

The whole limits of t, viz., from 0 to ∞, cannot be taken at once, as $\frac{dc}{dt}$ keeps reversing, and this reversal will not affect the luminous discharge. Referring to the curve (fig. 9) it will be seen that the first reversal must take place at M_1, when $bt=\theta$, and subsequent ones for values $\pi+\theta$, $2\pi+\theta$, &c., of bt. It is therefore necessary to take first the limits θ and 0, then $\pi+\theta$ and θ; $2\pi+\theta$ and $\pi+\theta$, and so on, alternately writing the integrals plus and minus.

$$\int_0^\infty \frac{dc}{dt}\,.\,dt=\frac{V}{bL}\left\{ {}_{0}^{\theta}\left[e^{at}\sin bt\right]-{}_{\theta}^{\pi+\theta}\left[e^{at}\sin bt\right]\right.$$
$$\left.+{}_{\pi+\theta}^{2\pi+\theta}\left[e^{at}\sin bt\right]-\text{\&c. ad inf.}\right\}.$$

Remembering that

$$\sin(\pi+\theta)=-\sin\theta,$$
$$\sin(2\pi+\theta)=\sin\theta,$$
$$\sin(3\pi+\theta)=-\sin\theta,\text{ and so on,}$$

this gives

$$\int_0^\infty \frac{dc}{dt}\cdot dt = \frac{2V}{bL}\sin\theta\left\{e^{\frac{a}{b}\theta} + e^{\frac{a}{b}(\pi+\theta)} + e^{\frac{a}{b}(2\pi+\theta)} + \&c.\right\}.$$

The series in the bracket is a geometrical progression, in which the constant factor is $e^{\frac{a}{b}\pi}$; and, since a is negative, this is less than unity.

Hence

$$\int_0^\infty \frac{dc}{dt}\,dt = \frac{2V\sin\theta}{bL}\cdot\frac{e^{\frac{a}{b}\theta}}{1-e^{\frac{a}{b}\pi}},$$

and

$$\tan\theta = \sqrt{\frac{4L-KR^2}{KR^2}},\ \text{or}\ \sin\theta = \sqrt{\frac{4L-KR^2}{4L}};$$

also

$$b = \sqrt{\frac{4L-KR^2}{4KL^2}},\ \text{and}\ \frac{\sin\theta}{b} = \sqrt{KL};$$

$$\therefore \int_0^\infty \frac{dc}{dt}\,dt = 2V\sqrt{\frac{K}{L}}\cdot\frac{e^{\frac{a}{b}\theta}}{1-e^{\frac{a}{b}\pi}}$$

$$= \frac{2V\sqrt{\frac{K}{L}}}{e^{\theta\sqrt{\frac{KR^2}{4L-KR^2}}} - e^{-(\pi-\theta)\sqrt{\frac{KR^2}{4L-KR^2}}}}.$$

Let $\theta\sqrt{\frac{KR^2}{4L-KR^2}} = x$ and $\pi\sqrt{\frac{KR^2}{4L-KR^2}} = y$.

Then the denominator $= e^x - e^{x-y}$, and

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \&c.,$$

$$e^{x-y} = 1 + x - y + \frac{x^2}{\underline{|2}} - \frac{2xy}{\underline{|2}} + \frac{y^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} - \frac{3x^2y}{\underline{|3}} + \frac{3xy^2}{\underline{|3}} - \frac{y^3}{\underline{|3}} + \&c.,$$

$$e^x - e^{x-y} = y + \frac{2xy}{\underline{|2}} - \frac{y^2}{\underline{|2}} + \frac{3x^2y}{\underline{|3}} - \frac{3xy^2}{\underline{|3}} + \frac{y^3}{\underline{|3}}$$

+ terms of the 4th, 5th, &c. powers.

Now x and y will in general be small fractions, since KR^2 is usually much less than $4L$.

If the oscillations are very rapid, θ is very nearly equal to $\frac{\pi}{2}$. Hence $y=2x$ approximately. Then e^x-e^{x-y} becomes

$$2x+\frac{2x^3}{6}=2x\left(1+\frac{x^2}{6}\right) \text{ approx.}$$

Therefore the time-integral

$$=\frac{2V\sqrt{\frac{K}{L}}}{2x\left(1+\frac{x^2}{6}\right)}=\frac{V}{x}\sqrt{\frac{K}{L}}\left(1-\frac{x^2}{6}\right) \text{ approx.,}$$

and

$$x=\frac{\pi}{2}\sqrt{\frac{KR^2}{4L-KR^2}}=\frac{\pi R}{4}\sqrt{\frac{K}{L}} \text{ approx.;}$$

so time-integral

$$=\frac{4V}{\pi R}\left(1-\frac{\pi^2R^2K}{96L}\right) \text{ approx.}$$

Now from this it is seen that the effect of increasing the capacity would be to slightly diminish the time-integral, and consequently probably make the brilliancy of the luminous discharge less, if it were not that increasing the capacity diminishes the real resistance of the circuit, since it makes the oscillations slower, and the resistance R for copper for rapid oscillations approximately equals $\sqrt{\frac{1}{2}blR_0}$; where l is the length of the wire, and R_0 its resistance for steady currents. Now $b=\frac{1}{\sqrt{KL}}$ approximately.

Therefore

$$R=\sqrt{\frac{lR_0}{2\sqrt{KL}}};$$

so that the time-integral is very roughly proportional to the fourth root of capacity.

There is also another reason why larger jars might produce a brighter discharge, even though the time-integral were less.

With larger jars the time taken for the amplitude of the current to sink to a value at which it becomes insignificant will be longer than in the case of small ones. Now, as the initial value of $\frac{dc}{dt}$ is the same whatever the size of the jars, the after values (although their time-integral is less and their actual values less also) last longer in the case of larger jars.

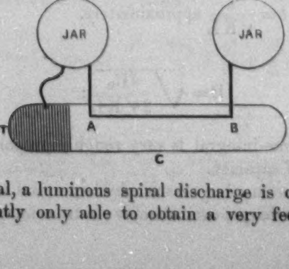
When the breakdown of the dielectric of rarefied gas is once begun by the initial $\frac{dc}{dt}$, the values of $\frac{dc}{dt}$ necessary to keep it up may probably be very much less, and consequently the smaller values of $\frac{dc}{dt}$ lasting longer, as given by the larger jars, may produce a luminous discharge more brilliant to the eye than the larger values of $\frac{dc}{dt}$ lasting a shorter time, as given by the smaller jars.

The actual results obtained with a ring of four turns of wire containing an exhausted bulb about 2½ inches in diameter were that the differences in brilliancy, obtained by using half-gallon jars, pint jars, or very small jars made from specimen glasses, were not so very great.

Other Effects. Apparently unclosed Discharges.

A closed luminous discharge is not the only one that can be obtained. Mr. Tesla, in 1891, pointed out that by wrapping a wire round an exhausted tube so as to form a coarse-pitched spiral, a luminous spiral discharge is obtained. He was apparently only able to obtain a very feebly luminous

Fig. 10.



spiral, but the author has succeeded in getting one quite as brilliant as in the case of the ring-shaped discharge obtained with a bulb.

In fig. 10 two half-gallon jars have their outer coatings connected by a wire, A B, bent as shown in the figure. Over the wire is laid an exhausted tube, C, with a tinfoil cap*, T, at one end; T is connected to the outer coating of the jar nearest to it. The object of this is to utilize the electrostatic effect and make the tube more sensitive to breakdown by the electromagnetic one. When the jars discharge, a straight luminous band is observed in the tube directly over A B.

If the tube C be now moved towards the jars, even by a very small amount, a closed circuit discharge will be obtained. There is apparently, then, a tendency for the luminous discharge to form a closed circuit whenever possible; and it seems probable that even when the discharge is apparently not closed, as in the case of the spiral or the straight line, the electric stress acting in the rarefied gas takes the form of a closed circuit, but is only intense enough to produce sharp luminosity close to the wire†. To further test the question an unclosed ring tube was made, and when it was placed inside a coil of wire no trace of a single luminous band could be seen‡. A small glass tube was also bent so as to form a spiral of four turns, and exhausted. A wire following the spiral was attached to it, but this also gave no trace of luminous discharge.

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* It is not always necessary to use this cap, as, if the exhaustion is high enough to give green phosphorescence of the glass, with the two half-gallon jars in series, the discharge can be obtained without the cap. With another tube of lower vacuum the author finds the cap necessary.

† That is, the return part of the discharge is so diffused and feebly luminous as to easily pass unnoticed in comparison with the sharp and brilliant luminosity close over the conductor. The same applies to the spiral discharge, each turn of the spiral probably forming a closed circuit by itself.

‡ On afterwards repeating this experiment the author obtained a discharge in parts of the tube, and with half-gallon jars in the whole tube. The discharge, however, was a closed one, as there were two distinct bands in the tube, one on the side next to the coil and the other on the side farthest away from it. This is what might be expected if the magnetic induction be sufficiently strong.

Magnetic Effects of Discharge.

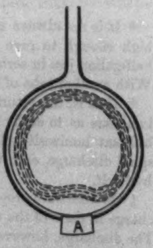
The ring discharge in a bulb or closed circular tube acts like a metallic circuit as far as magnetic effects are concerned. This may easily be shown by the following experiment.

A coil of three or four turns of wire has a similar one wound with it to form a secondary; the latter is connected to a third coil, in which is placed an exhausted bulb. The first coil is connected to the outside coatings of the jars (fig. 1). The spark-gap can be adjusted so that a fairly bright ring is produced in the bulb. If now a second bulb is placed within the first coil a luminous ring will be formed in it, and the ring in the other bulb will be much weakened or altogether extinguished. Exactly the same effect is produced if a metal plate or closed coil be brought near the first coil in lieu of the bulb.

Sensitive State of Discharge.

If a single turn of insulated wire surround one of the exhausted bulbs as in fig. 1, and the spark-gap be adjusted so as to produce a rather faint luminous ring (the fainter the better); on approaching the finger and touching the wire at any point the discharge appears to be repelled, and takes the shape shown in fig. 11. Instead of touching the wire with the finger a small piece of tinfoil may be laid between the wire and the bulb, as at A (fig. 11), and this may be touched by the finger or connected to any large object, insulated or otherwise; the effect produced is the same. Connecting the tinfoil to one of the outer coatings of the jars does not produce this effect, and it is scarcely, if at all, visible when the luminous ring is brilliant, due to a longer spark-gap. With a wire ring of several turns the author has not been able to obtain it. If a turn of bare wire be employed the effect is produced when the finger is brought very near to the wire, but if it be brought into

Fig. 11.



actual contact the effect is no longer visible. This apparently shows that it is due to the capacity between the finger or tinfoil and the wire; it is probably of the same nature as the "sensitive state" in an ordinary vacuum-tube.

ADDENDUM, May 1st, 1893.

Since writing the above the author has made a further experiment* which at first sight appears to contradict the one† given in the paragraph on "Magnetic Effects of Discharge."

A ring (R) of four turns of wire is joined in series with a single turn, and the two are connected to the outside coatings of the jars. In the single turn a bulb is placed and the spark-gap adjusted until a fairly bright ring is produced in it at every discharge. If now a closed ring of thick copper-wire, a metal plate, or a ring of several turns, similar to R, and with its ends joined, be laid on R to act as a secondary, the luminous ring in the bulb is brighter; on substituting for this an exhausted bulb and placing it in R, there will be a brilliant ring-discharge in it, while the discharge in the other bulb will be rendered fainter or altogether extinguished. In this experiment the exhausted bulb secondary appears to act in the reverse way to a metallic secondary.

The author then made the following experiments:—

(*a*) A ring of four turns of guttapercha-covered wire precisely similar to R was made, its ends were connected to an ordinary Geissler tube. When this was used as secondary it acted exactly in the same manner as the exhausted bulb both in the first and second experiments, the Geissler tube being brilliantly illuminated.

(*b*) The Geissler tube was then removed, and the ends of the secondary coil connected to the coatings of a small Leyden jar. The effects produced by this secondary were the same as those produced by the exhausted bulb in both experiments.

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* Called hereafter the second experiment. This experiment was shown when the paper was read.

† Called hereafter the first experiment.

(*c*) The ends of the secondary were connected to the loops of a glow-lamp to act as a resistance (about 100 ohms). This acted similarly to the exhausted bulb in both experiments.

(*d*) A disk of gilt paper (imitation) and also a ring of the same were used as secondaries; these acted similarly to the bulb in both experiments. When the discharge took place there were brilliant sparks produced at various spots on the paper, wherever there was any flaw in the gilding, showing that considerable energy was dissipated there.

(*e*) The secondary coil of four turns had its ends joined by a strip of gilt paper about 6 inches in length, with a considerable number of flaws in the gilding (produced purposely, by bending the paper sharply in several places, so as to obtain considerable sparking). This acted similarly to the bulb and dimmed the discharge in the bulb surrounded by the single turn. On shortening the length of gilt paper between the ends of the secondary, the discharge in the bulb was less dimmed.

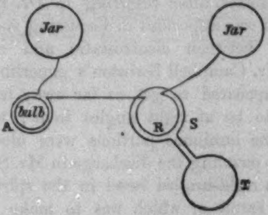
The results of these five experiments are, that any of the above secondaries are able to reduce the mutual induction between the primary and secondary in the first experiment sufficiently to render faint or altogether extinguish the discharge in the bulb, and act similarly to an exhausted bulb secondary. In the second experiment a low resistance secondary behaves in the reverse manner to an exhausted bulb secondary, while (*c*) and (*e*) show that a high resistance put externally into the secondary circuit, and (*d*) that a secondary having a high resistance in itself, act in a similar manner to an exhausted bulb secondary. (*b*) shows that if the ends of the secondary be attached to a capacity it behaves like the bulb.

The most probable explanation seems to be the following:—The amount of energy in the jars when charged is a fixed quantity for a given spark-gap; this energy will be mostly expended in the coil R and the single turn and bulb (the second experiment). If, now, we can make energy be expended elsewhere, as in a secondary, we shall have diminished the energy received by the bulb, and this will in general dim it or altogether extinguish it. This will explain what happens when an exhausted bulb secondary is used; also experiments (*a*), (*c*), (*d*), and (*e*). With regard to experiment (*b*), energy may have been expended in heating the glass of the jar on account of electric hysteresis. Moreover, this secondary did not dim the bulb so much as the others, but was found to be capable of improvement in this respect by including some resistance (in the shape of the glow-lamp or a strip of gilt paper) in its circuit.

In the case of a low-resistance secondary the energy dissipated in it will be small, since its impedance will not be much lessened by its being of low resistance on account of the high frequency. This does not explain, however, why the discharge in the bulb is brighter when a low-resistance secondary is used*.

A further experiment was then made. The coil R in the second experiment had a similar secondary S placed in it; this was connected to another similar coil T. The spark-gap was lengthened until a brilliant luminous ring was produced in a bulb placed in T. The bulb in A was then moved away from A until there was a very faint luminous ring in it. On

Fig. 12.



A one turn; S, R, and T each four turns.

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* This energy explanation is probably not a complete one. Working out the frequency in the cases of no secondary, a secondary of four turns short-circuited, and the same with its ends joined through 100^{ω}; the author finds that the damping-term is increased when either secondary is used, but more so with the 100^{ω} in circuit. The frequency is much the same with the 100^{ω} in circuit as when there is no secondary, but with the secondary short-circuited the frequency is about doubled. This may account for the increase in brightness of the discharge in the bulb.

removing the bulb from T a very slight brightening of the faint ring of the bulb in A was observed. Instead of placing an exhausted bulb in T, a coil of four turns with its ends joined through 100 was laid on T, and the bulb in A adjusted to give a very faint ring; on removing the coil from T a decided brightening of the discharge in the bulb was observed. This experiment seems to show fairly conclusively that increasing the energy in the circuit of the secondary S diminishes the brightness of the discharge in the bulb placed in A*.

DISCUSSION.

Dr. Sumpner, speaking of the apparently unclosed discharges, pointed out that they might be closed through the wire forming the primary circuit in the same way as a coil of a transformer might be arranged to act, partly as primary and partly as secondary.

Mr. A. P. Trotter, after referring to Dr. Bottomley's researches, said it was important in discussing such experiments to distinguish between electrostatic and electromagnetic effects. In Mr. Campbell Swinton's experiments the luminosity always appeared to get as far away from the wire as possible, and to be at right angles to it; whereas in Mr. Rimington's the luminous portions were close to the wire. With a view to puzzling the discharge in Mr. Swinton's tubes, he had made a right-angled bend in the spiral surrounding the tube, the result of which was to make the luminosity discontinuous, one end of the break being bifurcated. In all Mr. Swinton's experiments brush-discharges surrounded the wire.

Prof. S. P. Thompson thought an electrostatic field would aid a discharge even if its direction was not the same as the E.M.F. due to varying magnetic induction. Planté had found that vacuum-tubes through which 800 cells were insufficient to produce a discharge, immediately allowed a discharge to pass when a rubbed ebonite rod was brought within about ten feet distance. This effect was found to be independent of the direction of the disturbing field. Analogous effects had also been observed by Prof. Schuster, and described in his Bakerian Lecture.

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* Since writing the above the author finds that Prof. J. J. Thomson has observed the effects noticed in the second experiment, and gives an explanation practically identical with the above.

Mr. E. W. Smith regarded the stresses set up in the medium as cumulative, a very slight cause acting on a substance already strained nearly to breaking-point being sufficient to cause breakdown.

Mr. Blakesley inquired if the effects were the same if the induction-coil used in one of the experiments was replaced by an electric machine, and whether the direction of the field so produced influenced the result.

Mr. W. R. Pidgeon said closed circuits were necessary; he had found it very difficult to produce discharges in tubes unless the ends of the primary wire were brought together.

In his reply, Mr. Rimington said each turn of the luminous spiral formed a complete circuit of itself. The phenomena observed by Mr. Campbell Swinton were quite different to those he had shown and due to different causes. Mr. Swinton's spirals were reversed, and were due to phosphorescence of the glass.

PROCEEDINGS

OF

THE PHYSICAL SOCIETY

OF LONDON.

OCTOBER 1893.

XVIII. *The Foundations of Dynamics.* *By* OLIVER LODGE, *F.R.S., Professor of Physics in University College, Liverpool**.

PART I.

The Nature of Axioms.

IT is a matter of congratulation with me that a critic who has devoted so much thought to Newton's laws of motion and similar fundamental doctrines should have begun a discussion of my papers on the subject of Energy; and I shall willingly consider his objections in order to see what modifications, if any, should be made in my original statements. But Dr. MacGregor's temporary attitude towards Physical problems is exhibited rather strikingly in a treatise on "The Fundamental Hypotheses of Abstract Dynamics," which he published as a Presidential Address to a section of the Royal Society of Canada (Transactions 1892). Hence, before replying to his criticisms on my writings, as made in the February number of the Philosophical Magazine (vol. xxxv. p. 134), I should like to make a few general observations suggested by this other deliverance of his, so as to indicate what seem to me the rather different points of view from which we, or if not we some other writers, approach these fundamental doctrines of Mechanics and Physics. The difference in attitude may be briefly summarised thus :—Some philosophers seek to advance truth by detecting or inventing complications in what was apparently simple ; whereas others aim at making simple statements concerning things which are apparently or really complicated. A generalization like this is not one which will bear pressing into individual cases, but if it contains an element of truth it has reference to no personal detail, as it seems to me, but to a difference in type ; and I sometimes think that most minds, except those few of the very highest order who are above classification, may be said to fall into, or at least to lean toward, one or other of these categories*. Each type of mind performs its service, and each type has its appropriate danger.

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* Read May 12, 1893.

The detection of a real complication is a service to truth ; the invention of a needless complexity is a disservice and temporary obstruction. The reduction of apparently complex facts to a simple statement in commonplace language is, I believe, a service ; the over-simple and incomplete summary of what is really complex is not an equal service, but I do not perceive that it is likely to be any serious obstruction : it seems to me rather of the nature of a first approximation, which is often temporarily helpful.

When Ohm stated his law that current is proportional to E.M.F., he did not know that it was really true. It has turned out to be precisely true for copper and for sulphate of copper—the only substances for which it has been seriously tested ; but even if it had not been so accurate, its statement was a service, since it enabled half a century to walk in the light instead of in the dark. There is no evidence that it is accurately true for every variety of solid and liquid conductor, but by this time it is the fashion to assume its truth in ordinary simple cases. And rightly so, as it seems to me; the burden of proof rests now with the enterprising experimenter who can detect a flaw in it. His evidence will be listened to, but till it is forthcoming vague doubts can be legitimately ignored.

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* I see no reason in Dr. MacGregor's book on Dynamics for including him in the first category: it is his Presidential Address on the Laws of Motion that alone suggested it. I do not intend the classification as in any way offensive : I should think that Prof. Karl Pearson, for instance, would willingly enrol himself under the first head rather than under the second, judging by his 'Grammar of Science.' But very likely MacGregor has stated the laws of motion in their simplest conceivable form if attraction and repulsion across a distance are to be contemplated. That is the essential difference between us : he is willing to base Physics on action at a distance ; I am not. From the action-at-a-distance point of view his statements are in many respects admirable, especially those near the conclusion of his essay. The remarks in the text are intended to have only a general and impersonal application.

Take another example:—The characteristic equation of gases in the simple form $pv=RT$ has done good service, though it turns out to be untrue for every actual substance. Without it, however, we should have been unnecessarily floundering in the dark. Even now it is more used in dealing with gases than any other equation. The improved statement of Van der Waals, adding a term to p and subtracting one from v, was another distinct service, and enabled a mass of experimental evidence concerning the structure of liquids to be conveniently and simply summarised. In its turn, however, it has had to give place to more complex empirical statements, and the complete law has not yet appeared.

The examples I have chosen, one of a precise, the other of an approximate, simple statement, are not indeed of the nature of axioms; and it may be held that it would be unsatisfactory to base our axioms on such a tentative sort of footing.

And yet what other course is open? Truly axiomatic statements can only be effectively made concerning things of which the race has had a long course of experience,—things to which they have grown familiarly accustomed. If they can be actually proved, they are theorems, not axioms.

The setting forth of an axiom I regard as a kind of challenge, equivalent to the statement:—"Here is what seems to me to be a short summary of a universal truth; disprove it if you can. I cannot prove it, it is too simple and fundamental for proof, I can only adduce hundreds of instances where it holds. I have indeed critically examined a few special cases and never found it fail, but a single contrary instance will suffice to overthrow it; hence, though it be hard to prove, yet if not true its disproof should be easy: find that contrary instance if you can." If no disproof is forthcoming for a few generations, the axiom is likely to get accepted. Meanwhile its undeniable simplicity is a practical advantage, even though in the course of centuries a flaw or needful modification in its statement may be discovered.

This is the kind of basis on which such a law as that of the Conservation of Energy or the Conservation of Matter rests. That the perpetual motion is impossible, that matter is indestructible, that energy never diminishes in the act of transfer, all these must be regarded as generalizations based on a great series of experiments, some consciously directed to the upsetting of one or other of the laws if possible, some aimed at establishing them, but most of a non-contentious and collateral character. If we are challenged to produce direct evidence that in any given chemical reaction the mass of the reagents is unchanged, not only in the initial and final stages but at every stage of the process, the proof may be exceedingly difficult. Heat is liable to be developed which would interfere with delicate weighings, and the reaction challenged may be an explosive or otherwise inconvenient one. But we do not attempt the proof, we shift the burden on to the shoulders of the doubter and say to him, Disprove it if you can ; and so we practically say for all our axioms, and for all laws which are so simple and fundamental as to be hardly distinguishable from axioms.

Experiments are often made or adduced in support of a law as if they were part of its foundation : thus Newton tried experiments on impact before stating his third law, but the experiments did not really prove it with accuracy even for the particular case examined. All they could show was that there was nothing obviously wrong with it. He saw no reason for supposing it wrong, and so after consideration stated it as an axiom, to be hereafter challenged and found inaccurate if so the progress of experience turned out.

I should say that an axiom or fundamental physical law is a simple statement, suggested by familiar or easily ascertained facts, probable in itself, readily grasped, and not disproved or apparently liable to disproof throughout a long course of experience.

If a statement is capable of exact examination and verification, either by reasoning or by experiment, it is called a law, but not a fundamental law; *i. e.* it is no longer part of the foundation, it is supported on something else. If it has no support, except the absence of evidence against it, it is an axiom. Far be it from me to decry the use of experiments of verification. The necessity for them whenever feasible is conspicuous and universally admitted, and much ingenuity may be usefully spent upon them; but I do say that in time a theory can become established by processes other than direct experimental verification; and in fact that really valid and flawless experimental verification is frequently an impossibility.

An instructive example of the legitimate strength of a theory, even when opposed by apparent facts, is contained in an article by Lord Rayleigh in the Philosophical Magazine for March 1889, "On the History of the Doctrine of Radiant Energy."

It appears that W. Herschel conceived the idea that the radiation which excites the sensation of light and the radiation which produces heat in black bodies are essentially different; and this view, which was contrary to his original intuition, was based upon a crucial prismatic experiment, made for the purpose of ascertaining "whether the heat of the red rays is occasioned by the light of those rays" or not. A definite question, *answered by experiment in the negative.* On this Lord Rayleigh remarks:—

"I am disposed to think that it was this erroneous conclusion from experiment, more perhaps than preconceived views about caloric, that retarded progress in radiant heat for so many years. We are reminded of Darwin's saying that a bad observation is more mischievous than unsound theory. It would be interesting to inquire upon what grounds we now reject the plain answer which Herschel thought himself to have received from experiment. I do not recall a modern investigation in which the heat and light absorptions are proved to be equal for the various parts of the visible spectrum. Can it be that after all we have nothing but theory to oppose to Herschel's facts?"

Yes, that is all, and, as Lord Rayleigh well knows, it is amply enough. Whoever examines the facts again will do so not to substantiate our present theory but in the hope of upsetting it. Success is of course just conceivable, but, when it comes, there will be time enough to reopen the question. Lord Rayleigh's words may be distorted, and may even suggest false meanings to minds with a crooked turn in them; so may many of the apparent admissions about unprovableness in this paper of mine, but, whether it gives occasion to the enemy to blaspheme or not, it is true that a host of doctrines are believed because they form part of a consistent scheme rather than because of any seriously attempted, still less any really achieved, experimental proof. And to pull one of these neatly fitting blocks from its niche will demand the strength of more than one, of more than several, so-called crucial experiments. There comes a time indeed when the weight of experimental evidence suffices to uproot a portion of tightly fitting theory; but seldom, as I think, without some looseness or uneasiness being first detected, and never without a betrayal of rottenness at the root.

Why do Physicists deny that matter can be moved by mental power from a distance without physical mechanism? Why does modern science reject the whole of a certain class of miracles, in the teeth of an immense record of direct experimental evidence? Solely because these things do not fit in with such coherent views of the universe as they have at present been able to frame.

Why, again, do we accept a multitude of unverified statements, such as, that every portion of radiation, whether it be of light or of sound, is intrinsically energy, and must, if absorbed, result in heat; that every muscular contraction of an animal corresponds to the combustion of a portion of his food; that a given gas consists of particles of approximately specified size and weight travelling at a certain average speed; that a medium connects every pair of bodies which are perceived to exert force on each other; and so on? Solely because these things do fit into a coherent and self-consistent scheme of the universe.

Any scheme or doctrine *sufficiently* harmonious and consistent goes far thereby towards establishing itself as truth. So conspicuously is this the case, when one comes to reflect, that there are not wanting some who conjecture that by our thought we are, so to speak, *constructing*, or at least helping to construct, the cosmic scheme.

Some axioms the human race has now given up challenging, and by so abstaining has silently accepted as corresponding to the truth of things. Others it occasionally exercises its ingenuity in degrading or depreciating, not into untruths, but into special cases of a higher and super-sensuous generalization. Varieties of space are imagined, and mathematically treated, where more than one line can be drawn through a point parallel to a given line, where the shortest distance between two points is not straight, where the three angles of a triangle are not equal to half a revolution, where a closed surface is an incomplete boundary, and where more than three lines can be perpendicular to each other. These things are imagined, and for all I know they may in some occult fashion exist. To set bounds to the possibilities of the universe on the limited evidence of our few sense-organs would be absurd. But I say that any proof of their actual existence within our more developed ken rests with the experimentalist. As soon as facts are forthcoming which clearly and definitely are inexplicable on the basis of our present notions concerning space, I for one am willing to enlarge those notions and to contemplate provisionally whatever hypothesis suggests itself as most simple and plausible. Till then there is plenty of work for a physicist in interpreting, systematizing, and clarifying the facts of the universe, as it appeals to him through the agency of his ordinary three-dimensional senses, aided by his undimensional common-sense.

But I hold that with all these vague possibilities of ultimate development in front of him (not so vague but that they are in some sort conceivable, or at least tractable by reason), a natural philosopher need not confuse himself by endeavouring to complicate what is already transparently simple ; nor will he be wise to attempt an over-laborious scrutiny of his fundamental axioms ; for the more neatly and quietly he can lay his foundations the more time will he have for building the superstructure, and the more gorgeous he may hope to make it. By all means let him avoid a rotten or insecure element in his foundation. It must be as sound and strong as possible ; but his underground work need not be decorated with fanciful and laborious conceits, it may be as plain as it is substantial, and he may leave its edges rough in order to connection with structures as yet unbuilt and unimagined.

Of this plain and substantial character would I seek to keep the laws of motion. If a statement like the first law of motion cannot be made in simple and readily intelligible language, I should despair of Physics. In that case the Physics of the future could be little better than a barbarous jargon of technicalities. There are plenty of really difficult places where technical language and unfamiliar modes of thought are for the present essential. The developments of the superstructure erected during the present century are indeed now so stupendous that for myself I should be satisfied if, without appreciably adding to them, I could by consolidation and restatement remove the necessity for some of this artificiality, and so make their harmony and beauty more readily appreciated. But in order to do this a simple and unlaborious foundation is a necessity. I hope to try before long to display the bold outlines of the foundation already laid by men of surpassing genius, every unnecessary accretion being cleared away, and the whole simplified to the uttermost; and then, if the attempt be not too ambitious, I should wish to extend the process to some portion of the superstructure.

PART II.

The First and Third Laws of Motion.

So much for general preamble; now turning to Dr. MacGregor's address, we find that he objects to the Newtonian statement of the first law of motion on the same ground that Prof. Karl Pearson, Prof. Mach, Mr. Macaulay of King's College, Cambridge, and several others* have objected to it, on the ground, namely, that uniform motion is unintelligible or meaningless unless you specify its direction and velocity with reference to a set of axes. And directly you try to specify axes you get into difficulties, for, although a uniform translation is permissible to them, any rotation or any acceleration of the axes is fatal to a simple statement of the behaviour of a body acted on by no force. It is useless to say that the axes must be stationary, because one cannot define what that means; so the attempt is made to say that the axes must not rotate and must not be acted on by force; but this last condition is of no use unless they possess inertia; so the axes are sometimes supposed to be generated by particles of matter projected in different directions and subsequently free from force*; or else the axes are made of infinite mass so that no finite force may be able to affect them.

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* See for instance a correspondent in 'Nature,' vol. xxxvi. p. 366.

Now I hold that all such notions as axes of reference are artificial scaffolding, necessary for the numerical specification of a velocity, but not at all necessary for the apprehension of what is meant by a uniform velocity. It is in the specification of any absolute velocity that the difficulties cited about an origin and axes of reference legitimately occur. It is in fact impossible to specify the absolute velocity of anything, because we have literally no criterion of rest. We shall, I believe, hereafter find it convenient to postulate the Ether as a body absolutely at rest; but none of these physical or geometrical complications should enter into an axiomatic statement.

All that the first law asserts is that the motion of a body not acted on by force is uniform in magnitude and direction. There is no need to attempt the impossible and say what that magnitude and direction absolutely are. Whatever they are they remain constant. If asked to prove this statement, we should at once decline, and throw the burden of disproof on the doubter. This is what Maxwell does† when he says (virtually):—If the speed and direction of a freely moving body vary they must vary in some definite manner; very well, tell me in what manner they are varying. You cannot, unless you can show me absolutely fixed lines of reference.

The fact is that the conception of uniform motion is based upon a simple primary muscular sensation, or at any rate upon a succession of such sensations; everybody understands what it means, so far as it is possible to understand anything in this material universe, and the sense in which it is understood is amply sufficient as a basis for a physical superstructure. The first law is a true axiom, and its boldest and simplest form is not only the best, but is the only one that can with any justice be called axiomatic. How can one appeal to the experience of the human race with reference to coordinate axes of infinite or any other mass? How can we utilize as axes the trajectory of particles free from force, without tacitly assuming the first law continually? The whole attempt to complicate the statement of the first law of motion seems to me absurd.

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* Thomson and Tait, vol. i. Part I. (1879), § 249. But although these writers do propose to use such axes to fix direction, or, better, the invariable plane of a rotating system, §§ 267, 245, they quite logically deduce these things from Newton's laws, and do not use them in the statement of those aws.

† Matter and Motion, art. xli. p. 36.

The well-known other objection, that a statement of the first law is unnecessary because it is only a special case of the second, rests on a different footing. Thomson and Tait have pointed out that it acts as a definition of equal intervals of time. Prof. MacGregor denies that the first law gives us any more useful definition of time than the second does; but seeing that every clock is an attempt at a uniform mover, and that the second law is concerned with the more complex notion of acceleration rather than with the simple idea of velocity, I do not imagine that he will seriously adhere to this view; and therefore, fully admitting the obvious fact that the first law is a special case of the second, I still hold that its separate statement is desirable, because it is so simple, and because it does afford a clear practical definition of the mode of measuring time. But were this its intended meaning it could have been expressed more straightforwardly. Newton probably considered it as a qualitative statement introductory to his second law: and as such it is entirely suitable.

None of the objectors to the first law have the slightest doubt of its truth,—that is what makes their contentions so practically futile; it would appear that it is too simple to please them; they seem to wish to complicate its statement so as to make it look more like the difficult things with which they are accustomed to deal. I feel convinced that many mathematicians mistrust a simple statement in English, and hardly conceive that such a language can really express an important law; their trained ability to deal with difficult conceptions leads them, as I think, astray.

I am not quite clear what the word logomachy means, but much of the discussion which has been bandied about concerning the statement of the first law seems to me to be rightly designated by some such opprobrious or perhaps complimentary epithet.

When Dr. MacGregor goes on to consider the third law, and to deny that it can properly be regarded as a deduction from the first, he is urging a very minor matter, if what I have said concerning axioms has any truth in it. Still the question has some interest. Whether it is deducible from the first law or not may be held to depend on how general the terms are in which that law has been stated. If it can be axiomatically asserted that the *centre of mass of a rigid system* moves uniformly until an external force acts on the system, and also that the system does not begin to spin, then the third law is established. For since zero acceleration means zero force, it follows that all the internal forces add up to zero, and have no moment; and since the system can be dissected bit by bit without ceasing to be a system within the scope of the first law, it follows that no stress can contain an unbalanced force or couple.

If the law be doubted for the case of a pair of bodies attracting each other from a distance, Newton says* (virtually):—Jam the bodies apart with a rigid obstacle, then you have reduced their action to contact action; and since you have a balanced stress at each point of contact, and likewise between the ends of the introduced obstacle, it follows that the attracting forces of the distant bodies are also balanced.

Now Dr. MacGregor's objections are (1) that Newton's proof only aims at extending the law from contact action to actions across a distance, while for contact action he is content to assume it as an axiom or to verify it by experiment; and (2) that the proof breaks down for a particle or single body which cannot be analysed into parts.

It is quite possible that Newton thought it best to state his third law as an axiom; because the fact that the centre of mass of a complex system of bodies obeys the first law is hardly an experience that can be confidently appealed to, even though all the bodies are in contact. That fact and the third law are intimately *connected*, but whichever is the simpler had better be stated as the axiom, and the other be made a deduction from it. Thus the centre of mass statement follows from the third law, and so very likely Newton preferred to arrange it. It is frequently difficult to know which of two very simple statements is the more axiomatic; and methods of proof are notoriously susceptible of considerable variation. The important thing is to notice the link or tie between two facts, to show that they mutually strengthen each other, not to pretend that one is beneath the other and supports it.

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* *Principia, Scholium* to *Axiomata.*

Attempts to build even so simple a structure as geometry in the form of a single column, stone upon stone, have been found artificial and in the long run impracticable. An enlarged basis of direct appeal to experience is not only necessary but desirable, and all fundamental matters should be kept low down, as nearly in contact with first principles as possible.

The so-called deduction of the third law from the first or second is important as a clear and strong cross-connexion between the two things, and need not be considered as a rigorous proof. It is rigorous enough if the premisses are granted, but if not, then there is a certain outstanding axiomatic or unprovable character to be shared between them; but this outstanding portion is, by reason of the cross-connexion, so slight as to remove all difficulty from what otherwise would seem, and indeed, strange to say, to many still does seem, an exceedingly tough morsel to swallow.

[I mean that one constantly finds examination candidates, and even Engineers, when catechized about a horse pulling a cart, though they may, some of them, politely admit that approximately the pull of the horse and the pull of the cart are equal (constituting the stress in the trace whose inertia we may agree to neglect), yet nevertheless assert that in reality the pull of the horse must be *the least little bit* bigger than the pull back of the trace, else the thing could not start. The fact is that the universal truth of the third law is *not* axiomatic, or at least is not obvious*, and hence its deduction from the other two laws is really a useful deduction.]

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* See for instance 'Nature,' vol. xxxvii. p. 558; and see all recent volumes of 'The Engineer' *passim*, especially about 1885 and 1891.

I do not see any point in Dr. MacGregor's second objection that the proof is inapplicable to a body without parts. For if such a body anywhere exists, plainly its parts cannot act on each other, and so there are no actions or reactions in such a body worth troubling about.

The concluding portion of Prof. MacGregor's address has to do with the Conservation of Energy and the question of how far it can be deduced from the third law. But all these questions respecting energy we must fight out at greater length. Dr. MacGregor well knows that from the third law and the denial of action at distance together I claim to have deduced a law of conservation simpler and more precise than the ordinary law; but his objection to this* is that though such a law may come to be accepted as sufficient in the future, when the universality of contact-action is fully recognized, it is inadequate for the present, when action at a distance still holds a portion of the field; by which I suppose he really only means that many mathematical methods of treatment are based at present on action-at-a-distance modes of expression. I have no fault to find with any convenient mode of attacking specific problems; it is permissible to everyone to use the language of distance-action for practical purposes; but when it comes to formulating fundamental laws I have no ambition to legislate for such cases until they can be shown actually to occur. I am open to experimental proof of their existence, but to none other. It is premature to legislate for them. If any action other than contact-action exists, we had better know more about it before formulating its laws. If the will-power of a "medium" for instance can really move a chair without any kind of contact or physical connexion, it will be wise to look carefully for the seat of the other component of the stress, if there is one, and for the source of the energy concerned, but I myself should feel extremely hazy as to their probable locality.

When action at a distance does present itself in Nature (and if it ever does it is clearly going to be in connexion with the operations of Life), it will be very well to overhaul our axioms to see if they require modification. Till then I propose to state them in terms of the facts we know. This I will attempt in another Part.

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* Expressed elsewhere, viz. Phil. Mag. February 1893.

PART III.

The Conservation of Energy and Universal Contact-action.

The ground is now clear, I think, for a reply to Professor MacGregor's criticism, as made in the Phil. Mag. (vol. xxxv. p. 134) for February 1893; and incidentally I may hope to answer or at least discuss the matter with some other critics, notably Mr. Heaviside in his paper in the Phil. Trans. 1892.

The first objection is that in my definition of energy I assume the ordinary law of conservation, because I say, in an early paper (Phil. Mag. Oct. 1879, viii. p. 278), "Whenever work is done upon a body, an effect is produced in it which is found to increase the working-power of that body (by an amount not greater than the work done); hence this effect is called energy, and it is measured by the quantity of work done in producing it." "The words 'is found,'" says Dr. MacGregor, "indicate an appeal to experience." Most true, so they do. My position is this:—Before making any definition it is desirable and only civil to show the reasonableness of it. To thrust a statement out without preamble or explanation, under cover of the contention that being only a definition or an enunciation one is at liberty to define or enunciate as one pleases, is I fear a thing frequently done, but it is barely polite, and it is apt to excite either resentment or else undue and slavish submission.

It is from no lack of love for Cambridge and the great men she has nurtured that I venture to hold that the typically Cambridge plan of text-book is liable to err in this direction. An attitude of blind faith and mere assimilation is required of a student for many of the earlier chapters, sometimes for the whole of a book. The life, the interest, of the subject to be treated, an exhibition of the reasonableness of the adopted mode of treating it, are all neglected, and a ghastly skeleton is presented to you bone by bone; which truly is an admirable structure when subsequently clothed in flesh by men of sufficient genius, but which is liable to excite repulsion in a thinking student who possesses some physical instincts but is devoid of the artificial galvanic stimulus called Examination*.

It may be that I myself err in an opposite direction, being so interested in the muscles and the clothing that I forget here and there a bone or two. If it be so it is a grievous fault, and grievously shall I have to answer for it. It is not a fault that I ever attempt to justify. Flabby and boneless science is no science at all.

In defining Energy, then, I first appeal to experience that something in the popular sense *energetic* is often plainly produced in a body when work is done upon it; *e. g.* when a bow is stretched or a stone flung; and I proceed to say that that something is always produced, even when not obvious, and that it has been called Energy because it frequently confers upon the body possessing it the power of itself doing work. I know well enough that the common definition runs, Energy = power of doing work, but there is a difficulty about this definition: plenty of energy has no power of doing work, or at least no power that we can get hold of. Therefore I prefer to give a name to the result of work done, whether it be obviously energetic or not, and to justify the name energy by appealing to the many cases where its possession does confer a power of doing at any rate some work; but in the definition I make no statement that energy must necessarily be able to do work, or must necessarily continue constant in quantity, because such statements, so far as they are true, are part of the law which is being led up to, not part of the definition.

My definition of "energy" stands on all fours with the customary definition of the potential function. It is a name for the line-integral of a force, considered as a quantity that can be stored. The line-integral of a force in action is *work*, the result of it is *energy*. Or, otherwise:—The scalar product of force and velocity is *activity* (sometimes called power); the result of activity lasting a finite time is energy.

Some conception of what energy is like can be gained by appeals to familiar experience; and the fact that it can be stored more or less completely, and can ultimately reproduce an activity similar to that which generated it, can also be illustrated in a selection of cases. The law of energy asserts, what cannot be so readily demonstrated, that in *all* cases it can be stored without loss, except in so far as it leaks away, or until it is discharged, in some kind of equivalent activity; and that even when thus dispersed or transferred or lost to the body in which it was generated, it still exists in other bodies in undiminished quantity, and so is capable of transference and retransference, by means of activity, and activity alone, for ever.

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* Valuable enough, however, in its proper place. I am not joining in the outcry against examinations. They are a most useful and much needed stimulus to right intellectual conduct, and partially replace the old belief in purgatory.

But how is such a wide-sweeping statement to be justified? Not by any hasty appeals to experience, not even by the laborious researches of a Joule. Such experiments can indeed examine the obviously weak places and can show that there is no manifest flaw there, that all the apparent flaws disappear on closer inspection, and, so far as can be seen, do not exist. The law itself as a universal generalization must be itself axiomatic, or else must be deduced from some other and simpler axioms.

I hold that it can be deduced from Newton's third law and from the denial of action at a distance, thus:—Bodies can only act on one another while in contact, hence if they move they must move over the same distance *; but their action consists of a pair of equal opposite forces; therefore the works they do, or their activities, are equal and opposite; therefore, by definition (p. 303), whatever energy the one loses the other gains. In other words: in all cases of activity, energy is simply transferred from one body to another, without alteration in quantity.

I claim also that the law of conservation thus established is more precise than the ordinary law (which I confess always seems to me rather vague, especially when such absurdities as "possible" and "actual" energy are put into its statement,—a denier of the law could use no more deadly word than "possible"); more precise and definite, I say, because it is the law not only of *conservation* but of *identity*. I believe that questions arising from this law of the identity of energy—a study of the paths by which any given now-existing bit of energy has reached its present locality, and of all that has happened to it in the past—may prove in the future to be as fruitful a region of enquiry as are studies in the history of any given piece of matter, say the earth or the sun, or, to step forward a little, the history of an individual mind, when we realise some day that this, too, has a continuous, and perhaps traceable, existence.

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* Motion in the line of stress is here directly contemplated. As has been already several times noticed, the possible *slipping* of bodies in contact is not apparently allowed for, but in so far as they are smooth their slip is ineffective, and if they are rough there is a tangential stress to be taken into account as well as a normal pressure; the friction-stress is intermittent instead of steady, and molecular instead of molar, but otherwise the above statement applies without complication, so far as it ever applies to the immediate action of ordinary material bodies.

However, Prof. MacGregor objects (p. 135) that my deduction of the conservation of energy proves it during *transfer* merely, and leaves untouched the question of what happens during simple *storage*.

That is true, and I do not think that I had noticed the objection before. My deduction proves that in all cases of activity energy passes on without loss; it does not prove that without any activity energy may not leak away in some silent unobtrusive fashion, leaving no trace, but simply vanishing.

To meet this objection, an appeal may be made to the definition; it may be called on to assert that anything that can thus disappear without force or motion is not energy, according to its terms; but I doubt whether such an appeal is perfectly cogent, though it has some, perhaps much, plausibility. I think that the non-disappearance of energy in this occult manner had perhaps better be regarded as an axiom, a statement which may be believed until some clear experimental disproof is forthcoming: no such disproof being meanwhile in the least expected.

Briefly summarised, the matter stands thus :—

(1) If energy can only be got rid of by activity, and (2) if that activity never affects its quantity, the law of conservation is completely stated. My proof covers (2), but not (1). I may therefore agree with what I suppose Prof. MacGregor to mean, that the portion (1) must be left as an axiom based on experience.

Prof. MacGregor goes on to say (p. 137) that my law of conservation during mutual action has no deeper meaning than the conservation of their joint momentum, and is quite consistent with the non-conservation of their working-power.

To this I reply, first, that I mistrust the vague term "working-power," it is apt to mean whatever may be convenient; from one point of view a given amount of energy may have an infinite "working-power," since it can do work at every transfer without itself diminishing; while from another point of view it is rather bold to maintain the conservation of working-power in face of the doctrine of the dissipation of energy. And secondly I reply, that the conservation of *momentum* rests on the equality of the forces exerted by two mutually operative bodies, combined with the obvious equality of the *durations* or times of action of these forces. Whereas the conservation of *energy* rests on the equality of the forces, combined with the altogether less obvious fact of the equality of *velocities*, or *distances traversed*, by each of two operative bodies. Anyone in his senses who believes in action at a distance would deny that the velocities of two directly acting bodies are necessarily equal (no other bodies being assumed present, so as to simplify the problem); and in so doing he practically denies the conservation of energy.

Energy is only really conserved under conditions of universal contact-action.

Of course the believer in action at a distance is not at a loss: in order to retain his fiction he has invented another unreality, which he calls possible energy. He says (truly enough for many purposes) that when a stone is raised from the earth a great deal more work is done on the stone than on the earth; and hence that, although at every instant the stone and earth have equal opposite momenta, the energy is nearly all possessed by the raised stone. But since the energy of an inert mass is by no means apparent, since it only has the power of gaining actual energy when let drop, its energy when inert and merely elevated is called possible or potential. Or sometimes, rather more accurately but much more vaguely, energy is said to belong to the *configuration* of earth and stone[*].

Now this idea of potential energy is convenient as a mathematical expression, and used in its proper signification it is an essential reality; but used in the sense above quoted, a very common mode of using it, it is a mere receptacle for stowing away any portion of energy which it is not convenient for the moment to attend to; and I defy anyone to realize it as a thing.

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[*] *Cf.* Phil. Mag. June 1881, p. 532, and June 1885, xix. p. 484.

Yet I myself constantly employ the term potential or static energy, and assert that every activity not only transfers energy from one body to another, but also transforms it from kinetic to potential, or *vice versâ*.

Yes, but I also assert that the transformation can only accompany transference, and that transference cannot occur without transformation. Whereas, on the ordinary view, the energy of the raised stone is supposed to gradually transform itself into kinetic as the stone drops, but to remain in the stone all the time.

I say that the energy was no more in the stone when merely elevated than it is in a strung arrow or the bullet of a cartridge. The energy is in the bow, or in the powder, and is rightly said to be in a static, or strain, or potential form. It can transfer itself to the projectile, and simultaneously transform itself into kinetic, at the pull of a trigger *.

It is easy to express oneself so as to be understood to object to the whole idea of potential energy. It is not to the use of the term that I object, nor even to its misuse, so long as it be done in the temporary interest of some specific problem. It is by no means necessary always to attend to everything: acting mechanism is frequently, with convenience and brevity, ignored, especially when one is really ignorant of its dynamical nature; but in laying foundations we should not make these omissions and slurrings. It is the erroneous localization of energy in a fundamental or theoretical treatment of the subject that I deprecate, and I believe that everyone will agree that in all such treatment convenient fictions are better avoided.

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* On page 141, Prof. MacGregor makes a statement in which I entirely fail to catch his meaning. It runs as follows:—"When Prof. Lodge states that 'a bullet fired upwards gradually transfers its undissipated energy to the gravitation medium, transforming it at the same time into potential,' he seems to me to assume that the bullet is rigid and that the medium is without inertia." Surely my critic does not consider that either plasticity in a projectile or inertia in a gravitating medium is essentially involved in determining the height attained by a body thrown up.

There certainly exists potential energy in the case of a raised stone, but it does not belong to the stone; it belongs to the medium, whatever it is, which is exerting force on the stone and on the earth, and pressing the two lumps of matter together. It is properly called static, as distinguished from kinetic, because it is the energy of force, not the energy of motion *. As I have frequently pointed out, the two forms of energy correspond to the two factors of the product work or activity. Both factors are necessary for the actual performance of work. Until both factors are present, the energy s merely stored. As soon as the missing factor is supplied, it is *transferred* from the body acting to the body acted upon, *e. g.*, from the gravitation medium to the stone.

But it will be said, suppose the stone is not allowed to drop freely, but is used as a clock-weight; it is doing work all the time it slowly falls, hence when raised it must have had energy. But in this case I should deny that the stone is doing anything active; it is a necessary concomitant, it is a link of communication, it is not really itself doing work. Its presence indeed and that of the earth are necessary to the existence of the stress in the ether [permitting that or some other equivalent hypothesis for the moment]; but it is the ether stress which is doing the work and driving the wheels of the clock. A clock-weight has somewhat the same function as the string of a bow; it was the convenient means of generating the potential energy, and it is the means of communicating it to the thing to be driven, but it is not itself energetic, any more than the chain and ratchet-wheel which helped to wind it up were energetic. Or, to take another example, it is like the piston of an air-gun: without it the gas would not have been compressed, and it might be (though it is not usually) employed to let the pressure reversibly down again, but it is not itself an active agent either in the charge or the discharge. It is like a Holtz machine or a dynamo: needful to the charging of Leyden jars or storage-cells, but not in itself doing the work, either when used for charging or, when running as a motor, for discharging.

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* I do not mean to assert or deny anything concerning the nature of gravity. It may be that all energy is *ultimately* of the nature of motion, as we know Lord Kelvin has taught us may be the case with elastic stress for instance; but *force* is the proximate mode of action of a coiled spring, and so it may properly be described as static energy, whatever its ultimate nature. The reasons for affixing to static energy the epithet potential are mainly mathematical and historical. There is much convenience in thus linking it on to the potential function, and there is no particular inconvenience unless the adjective be misunderstood in the sense of "possible." It is indeed possible *work* or possible *activity*, but so is every kind of available energy; it is not possible *energy*, for that term is equivalent to the assertion that not yet is it energy at all.

I fear I am explaining elementary matters at some length; but, although elementary in one sense, they are not quite easy to seize by the right end; and the fact that they are so elementary tends actually to retard their apprehension, for unless people have patience to think them out they will not grasp what is meant. The difficulty in all these matters is that everybody thinks he understands them already, and is quite satisfied with his own prehistoric way of regarding them. Now I want to say that I have thought these things out with some care and labour; and I believe that although no doubt everybody does understand them sufficiently for practical purposes, yet, if anyone has faith and patience enough to consent to reconsider them from my point of view (assuming, of course, as in most cases I safely may, that it is not his already), he will sooner or later realize the advantage of it. This belief may be presumptuous, but if so I am willing to presume to that extent.

With the substance of the following quotation from Professor MacGregor, on p. 138, February Phil. Mag., I entirely agree.

"If there be actions in nature which are not actions at constant distance" [*i. e.* practically contact-actions] "Prof. Lodge's law is not applicable to them, while the ordinary law is. Even if it be admitted that all actions in nature are contact-actions, there are many groups of phenomena which, in the present state of our knowledge of them, cannot be investigated on the hypothesis of contact-action. The early stages of their investigation must be conducted by the aid of the fiction of action at a distance; and in such stages Prof.

Lodge's law is not applicable, while the ordinary law is. Hence Prof. Lodge's law is not so general in its applicability as the ordinary law."

I might prefer to express it differently: I would say that it is no discredit to the true law not to lend itself to fictions; but I would also say, and freely admit, that it is often permissible for practical purposes to work in a fictitious or incomplete manner (as, for instance, when dealing with the diurnal motion of the sun or the libration of the moon), ignoring communicating mechanism in cases where one is ignorant of or need not attend to its nature, and in fact proceeding as the great physicists have constantly proceeded when engaged on a practical quest.

Never have I imagined that the practice of men of genius is other than the practice most suited to their immediate end. But when it comes to the philosophy and essence of the matter, I do think that there is some fear lest a constant habit of ignoring unknown essentials for practical and temporary convenience should lead these great mathematicians into forgetting that they have ignored anything, should in the long run delude them into treating their fictions as if they corresponded to the reality of things; and it is permissible for me or others, as lookers-on seeing the structure from another point of view, unobtrusively to point out to the master builders, in their moments of leisure, the fact that while they have been engaged in the upper stories the foundations have almost automatically expanded, that the available basis of axiomatic reality has become broader and simpler than could be perceived when they started on their work, and that now some portion of their temporary scaffolding and underpinning may without danger, and with advantage to the general appreciation of the fabric, be discreetly and quietly removed.

PART IV.

The Dissipation of Energy, the Nature of Potential Energy, and the Second Law of Thermodynamics.

Passing on to Prof. MacGregor's fourth objection (p. 140) that I have not proved that transformation and transference necessarily go together, I hope to be able to meet him more nearly than has been possible to me on most of the other counts of his indictment.

My proposition was that the change of form is always from kinetic to potential or *vice versâ.* But it is not necessary to attach the same importance to this law as an ultimate fact, because it is so extremely probable that most or all cases of potential energy will ultimately be resolved into essential motion. Nevertheless it is for the present a convenient statement of truth, having the same sort of validity that the second law of Thermodynamics possesses.

The following is my demonstration of the law of transference and transformation: being indeed the detailed statement of the essential phenomena accompanying transfer of energy, on the hypothesis of contact-action.

A body or other medium exerting force at any point, and there moving in the sense of the force it is exerting, is in a state of activity: it is passing energy on ahead. If its speed continues constant, the force just spoken of cannot be the sole force, the resultant force on it must be zero; in that case it is a mere transmitter, not itself active, only passing on what it receives. But if it is itself active,* i. e. *parting with its own energy, then its speed at the acting point (the place of application of the force) must either decrease or increase.*

If its speed decreases, it must be parting with kinetic energy; if its speed increases, it is parting with potential energy.

Contrariwise :—

A body or other medium exerting force and being moved in the sense opposed to that force is receiving energy.

If it is not merely passing it on, in which case the resultant force acting on it is zero, its velocity must vary.

If the velocity increases it is gaining kinetic energy; if its velocity decreases it is gaining potential.

Now the two bodies or things here spoken of are necessarily existent in every case of activity; one is the agent the other the patient, one the emitter the other the receiver, one the acter the other the reacter, one the driver the other the driven ; and they are in contact while the activity lasts

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* Meaning by "a body" a thing of constant mass, an identical lump of matter. A thing of variable mass, like a rain-drop or a railway-train or a layer of moving gas, is gaining or losing matter as well as energy; and cannot be regarded as one simple *body.*

Being in contact, their velocities along the line of stress must increase or decrease together.

If their common velocity is decreasing, then the driver loses kinetic and the driven gains potential.

If their common velocity is increasing, the driver loses potential, the driven gains kinetic.

This is my proof of the necessary concomitance of transfer and transformation, and of the alternation of transformations. But it will be well to illustrate the matter further.

Let me first explain how I define potential or static energy as used in the proposition criticised. I mean by it simply the energy of a body under stress; an elastic body which is exerting force may always be said to have potential energy, notwithstanding that when the force comes to be analysed it may be perceived to result from some kind of motion. Thus a reservoir of compressed air is a store of potential energy, though it is well known that all the pressure is due to the motion of unstressed particles, and is really just as much kinetic as any of the energy of planets and comets. A powder-magazine is a store of potential energy, though quite possibly the forces awaiting a trigger to liberate them into activity are due to atomic or etherial motions; and the same may be said of a charged Leyden jar and of a bent bow.

Briefly the idea of potential energy corresponds to the idea of elastic force when the cause of that force is not sought. The idea of kinetic energy corresponds to the idea of sustained motion. Each corresponds to one of the factors in the product "activity," or **Fv**.

In order that a body may possess energy it must be *capable* of exerting force and also of moving ; but it need be *doing* only one of these things, and so long as it merely stores energy it *must* be doing only one of them. Both factors must concur before it can be active, both factors must be possible before it can be said to possess energy. The only question is, which factor does it possess meanwhile ? If it is exerting force, but stationary, then its energy is potential or static ; if it is moving quite freely, then its energy is kinetic

So long as the other factor is absent no activity manifests itself, no work is done, and the energy is merely stored. Directly the other factor is supplied transference begins, energy leaves the body storing it, and passes to the body supplying the other factor.

Now to make a *moving* body do work, a resisting obstacle must be supplied, and this second body thus exerting force receives a due proportion of the energy; and it receives it by reason of the force it exerts, that is, it receives static or potential energy from a body which possessed kinetic.

To make a *strained* body do work, motion must be permitted; the thing which is moved gains energy, and it gains it in the kinetic form from a body possessing potential.

Of course in both cases the body immediately receiving the energy need not retain it, but may rapidly or instantaneously pass it on to something else—in which case it resumes its first form; by simple alternation of transformations.

Also it is not essential that the potential energy of the particles of a spring shall be communicated to any *foreign* body: if it is transferred to the spring as a whole, considered as a mass possessing inertia, that is sufficient; in other words, it may be transferred from one set of particles to another set of particles, all forming part of what we usually speak of as a single body*.

To take one of the examples of Prof. MacGregor, viz. a loaded air-gun with its muzzle plugged, so that its bullet or wad is not allowed to be shot. The compressed air has potential energy; on its release its energy is transferred to the moving wad, which instantaneously hands it on to the air near the muzzle, compressing it, and thus retransforming itself into the potential form.

If it be objected that the energy never at any instant all existed in the kinetic form, the answer is no, but it necessarily passed through that form; it could only be by the intervention of motion that the energy of the air at one end was transferred to the air at the other end. The wad itself is unnecessary and may be dematerialized; the only thing moving may be the gas atoms themselves, but the atoms which are rushing are not the same as those under strain; the moving atoms are precisely those which have escaped from strain. As everybody knows, where the rush of gas is greatest in a constricted pipe there the pressure is least. Air rushing from one reservoir into another illustrates the law precisely, though I choose it as a case at first sight favourable to Prof. MacGregor's contention; in the narrow part of the pipe air is flowing against a higher pressure in front of it, it is compressing the air ahead of it by reason of its own momentum; the energy of each molecule is availably kinetic to precisely the extent to which it ceases to be potential, and any residual potential energy which does not become momentarily kinetic, and kinetic in the available sense of a windy rush, has not been transferred at all, but remains simply stored, as it would have been had nothing been released and no activity been manifested.

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* This sentence is true in one sense, but it is not a final or complete statement, and in a later paper a modification will be introduced.

If the two reservoirs (or two Leyden jars) are for instance equal in capacity, and if one had been originally empty, then one quarter the original energy remains still potential in the originally full vessel, another quarter has been transferred by the rush of gas into the originally empty vessel, and two quarters have been dissipated, so to speak, converted into heat by friction. If there is no friction, then this half of the original energy remains alternating for ever from the kinetic to the potential form, like the bob of a pendulum, and transferring itself at every half swing from the one vessel to the other.

Lest this last statement should permit misconception, it may be necessary to state explicitly that the energy is only hazily attributed to the vessel—the energy belongs of course to the gas molecules—and, further, that the molecules which are in motion are not the same as those which are under strain. The energy, that is the portion of energy concerned with the particular amount of activity contemplated, is continuously leaving the molecules recovering from strain and transferring itself to those in accelerated motion, and as continuously leaving those in retarded motion and transferring itself to those acquiring strain*.

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* See, however, the footnote on preceding page.

Further to avoid misconception it is necessary to admit distinctly that in speaking of the pressure-energy of a mass of gas we are subjecting ourselves to an unnecessary though convenient limitation. It is chosen as an example just *because* so much is known about it; it would have been easy to choose elastic solid examples, where the elastic stress is not so readily analysable, and where the idea of potential energy and the statistical grouping of a mass of molecules was not optional but necessary, in our present state of ignorance. I know well enough that the pressure of gas in a reservoir is really due to the motion of the particles, and I am willing freely to contemplate the case of Davy's "repulsive motion," no real contact or elastic impact between particles ever occurring, but only a rapid swing or asymptotic orbit round their common centre of gravity; but this rapid centripetal swing of particles getting within each other's molecular range is at present essentially an unknown process, and has to be relegated to the sphere of potential energy quite as much as in the not perhaps really dissimilar cases of elastic impact and gravitation.

Whenever an atom after collision retains its energy intact, there is no need to say that there has been either transference or transformation; mere retention or storage is sufficient for practical purposes (though if we attend to the details of a collision we find there really is always a double transference and a double transformation *); and if this is typical of what happens in every atomic encounter, whether it be with the walls of the vessel or with another free molecule, then, so long as the air remains compressed, there is merely a storage of energy, each molecule possessing and retaining its own share. But directly an opening is made, or a communication established with an empty vessel, it is equivalent to the removal of a portion of wall, and some of the atoms, with their energy, pass along without obstruction into a new region of space.

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* *E.g.*, suppose a quick molecule A strikes or otherwise collides with a slow but similar molecule B. The mechanism of the collision, being unknown, is unimportant, and is most conveniently thought of in terms of elasticity. A and B undergo distortion as they approach, and a certain portion of A's energy is communicated to whatever medium it is which keeps them asunder. This medium by its recoil then drives them apart, sending off B with the energy which A originally had, and transferring to A the small portion received from B. [This matter is discussed at greater length in Appendix 2.] Any part of the energy which was not transformed is also not transferred. Everything transferred from A to B necessarily underwent rapidly a double transformation. When other occurrences are analysed it will be found that the above is typical of what constantly happens. Yet it is not always necessary to attend to all details, and in the text above I speak as if A and B passed through each other, each retaining its original energy.

In one sense each may still be said to retain its original energy; but in order to regard it thus we must be capable of dealing with the atoms individually. If the atoms escaping from confinement are lost in the vastness of space, their energy is unavailable, and their pressure, on which entirely depends the potential energy of a compressed mass of gas, is *nil*. If they escape into another equal vessel, then only the first individuals can enter freely without obstruction, all the others experience encounters on the way, as if the hole in the wall were gradually being restored; and though at the end of the operation, when everything has finally settled down, each atom may be said to possess its original energy, yet the pressure, on which depends the potential energy of the compressed gas, has been halved, and the other half is unavailable or has been dissipated.

Plainly the idea of potential energy belongs to the temporary order of ideas, to which the dissipation of energy and the second law of thermodynamics belong, and is appropriate to the present period when we have not yet learnt how to deal with molecules individually.

The kinetic energy of a set of gas molecules is only really available when it is combined with momentum or angular momentum, *i. e.* when the motion of a majority of molecules has the same sign, when there are more plus terms than minus in either their translational or their angular velocity; in other words, when there is a wind or cyclone *.

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* It may be worth while to point out that the reason why momentum is necessarily preserved in cases of impact, while translatory energy is usually not conserved, is because momentum depends on the first power of velocity, and therefore is unconcerned with vibrations; while every kind of vibration, whether it be of sound or of heat, enters into the sum total of a thing depending on an even power of velocity, like energy, whose translational value is therefore to that extent diminished by the production of vibratory disturbances.

A stream of particles can be utilised, as by a windmill or a Pelton water-wheel, and that is an example of available kinetic energy. But usually the energy of fluids can only be utilised in the potential form; as in a hydraulic-press or steam-boiler.

To say that all potential energy will turn out to be really kinetic may be true enough, but it is not at present a specially helpful truth, for even in cases (such as compressed air) when we actually *know* it to be kinetic, we are bound for all practical purposes to treat it as potential, since the available working-power of a spring, or of compressed air, of gunpowder, of Leyden jars, and of many other things, depends on their potential energy alone; and such other energy as they may possess is conveniently ignored, as not concerned in any practical transformations or activities which we can bring about: it remains as an untransferable, and therefore inactive or useless residuum.

This is the real meaning of available energy, it is the portion concerned in transferences, the portion which can be transformed, *the portion which is able to transfer itself*. Potential energy, as commonly spoken of, is always of this character. An unwound watch-spring has lost its potential energy, it retains a quantity of untransferable heat-energy. A hot body cooling is in the same case; so long as it is hot, some of its heat-energy is transferable; and the transferable portion may well be regarded as and styled potential.

Available or potential heat-energy readily transfers itself to space or to other bodies, just as compressed air readily leaks out of its reservoir; and in so doing becomes less potential*. Even if it is all received by other bodies in the room, the potential energy of the contents of the room is to the extent of the leakage diminished.

To say that heat-energy is constant in quantity is the same as saying that the molecules of an escaping gas need not part with their energy but may individually retain it. In the abstract either proposition is true, but just as it was only the compression part of gas energy that was potential or available for us, so also it is only the high-temperature part of heat-energy that is similarly available.

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* It is tempting thus to use the adjective "potential" in the sense of available. I do not at present wish to justify this secondary usage of the word, but I let it stand as a suggestive and harmless eccentricity.

It is customary and correct to say that the available part of heat-energy is converted into other forms when allowed to be active and do work. A distinction is thus apparently drawn between heat-engines, on the one hand, and water-engines, compressed-air engines, or electric engines, on the other; because in them the water, the air, and the electricity flow away undiminished in quantity and only at a lower level, pressure, or potential.

This distinction is the cause of some confusion or bewilderment, and I verily believe of some incredulity. Prof. Osborne Reynolds has shown, in his interesting biography of Joule, how great were the difficulties felt by the most eminent men in realising that heat actually disappeared in a heat-engine: that less heat was given to condenser than was received from boiler; that not only temperature fell but *also* heat was lost; and we all know how Carnot's theory was based on the contrary hypothesis.

But one mode of avoiding what is reasonably felt to be an artificial and puzzling distinction, is to say that in *all* the engines mentioned transformation of available energy occurs as soon as activity begins, and that that engine is perfect which enables all the energy available to transform itself in the desired way.

The potential energy of the raised water (so to speak for the moment), of the compressed air, of the separated chemicals, disappears, and transforms itself into the motion of a turbine, a bullet, or a motor-dynamo; the potential energy of the hot "working-substance" disappears in like manner, and results in the motion of fly-wheel belts and shafting*.

It is inconvenient to speak of the energy of heat as kinetic. It *is* much of it kinetic, just as the energy of compressed air and everything else may be kinetic; but to us here and now its available portion is not kinetic, but potential. When we can deal with molecules it can be regarded how we please—alternately kinetic and potential, probably, like the energy of a vibrating fork, or on the average half-and-half, like waves;—but till then the kinetic energy of individual atoms is useless, it is the average energy of a group which alone is useful, and this is all that we attempt to utilise in every one of the cases cited. Practically useful kinetic molecular energy exists only when all the molecules are rushing one way, and that is never the case with heat.

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* Prof. Fitzgerald has pointed out to me that this analogy will work well if I use *entropy* as the analogue of the water in a waterfall, because entropy does really fall in temperature and remain constant in quantity while a perfect heat-engine is working. This suggestion I hope to develop along with other thermodynamic matters in a future paper.

By the energy of a spring we mean its energy over and above its useless energy of average temperature; by the energy of a storage-battery we mean the portion corresponding to the reduced lead and the peroxide; by the energy of a waterfall we do not mean to include the warmth of the water above absolute zero; we always mean that portion of its total energy which we aim at utilising, and so speaking we say that a watch, or a dynamo, or a turbine, are efficient machines. By the energy of a hot body we ought to mean that portion of its energy which it has over and above its useless energy of average temperature, the portion which it is willing and able to part with, the only portion we hope to use or aim at using; and so speaking we might call a heat-engine an efficient machine.

We get a notion of low efficiency in the one case, of high efficiency in the others—we confuse ourselves sometimes with statements about the dissipation of energy—all because we perversely *attend* to the energy of average temperature in the one case but not in any of the others. We have done it naturally enough, because in that one case our attention was specially directed to the subject of heat; but it may be a help to realise that all the cases are essentially similar and on the same footing.

True a steam-engine and boiler is *not* an efficient arrangement, only about 8 or 9 per cent. at the best, but that is because of the great unnecessary drop of temperature between furnace and boiler; starting with the temperature of the *boiler*, and ignoring all energy below the temperature of the condenser, it may be efficient enough, 80 or 90 per cent. I suppose. It is better not to pretend to be able to use average molecular energy until we have learnt how to do it.

The portion of heat which can at a single operation be converted into work is very nearly the same as that which leaks away when the body is allowed to cool; just as it is the potential portion which disappears from a standing Leyden jar, or a running-down weight, or a rusting spring, or a leaky reservoir. The difference between the cases is that whereas the capacity of Leyden jars and tanks is constant, capacity for heat is apt to vary with the other conditions of a body; hence intrinsic energy is not solely a function of temperature, but subordinately of volume also. This fact necessitates caution before the above statement can be regarded as complete.

What a body will freely yield throughout a cycle of operations, that can be utilised. We cannot advantageously gain energy by pumping it. Whatever must be pumped is unavailable. We require an artesian well or automatic supply of energy if we are to get work out of it.

A further statement also is necessary if we are to concern ourselves with practical modes of utilising the theoretically available energy; some such statement as the following:—Available energy can only be continually utilised by means of reversible operations. Unless the working-substance is restored to its original condition the process cannot continually go on; and any operation which is not reversible involves dissipation of energy or needless waste of availability.

That the available portion of heat should bear to the whole heat the same ratio which the available drop of temperature bears to the absolute temperature, is essentially but a definition of temperature; it is an assertion that temperature is best measured as proportional to heat, or that the zero of temperature may conveniently be taken to correspond with the zero of heat.

An essential part of the second law of Thermodynamics therefore proceeds to state itself in a very general and purely commonsense form, thus:—

The portion of energy which a body can automatically part with is alone available for doing work; and only that portion which is parted with reversibly is actually utilised.

There is no need to mention "heat;" it is equally true of every form of energy. When a cell has run down, or a reservoir leaked itself empty, as empty as it wants to, any further energy it may have is useless; and any portion which flowed out in an uncontrolled or irreversible manner will have been wasted.

There is something specific to be said about each form of energy in order to apply the above statement definitely to that form; and in the case of heat the supplementary statement needed is that heat will not automatically leave a body for others at higher temperature: if it goes to a hotter body it must be carried by matter, or electricity, or something else, so that it is not a pure and simple flow of heat. In other words, heat will not flow "uphill" by pure conduction; and conduction is the only mode of automatic conveyance of heat as heat*. Water, air, or electricity can flow "uphill" for a time and can do work at the same time, by reason of their property of inertia. Heat cannot: it has no inertia. None of the uphill processes can go on continually or cyclically†.

The law of dissipation of energy states itself thus:—

If a body has any portion of energy in such condition that it is able irreversibly to leave the body, that portion usually does leave, sooner or later. This is only a rewording of the customary statement that the potential energy of a system tends towards a minimum; or, really, except that circumstances often delay the consummation unpractically long, towards zero. The universe will be stagnant, though by no means stationary, when its potential energy is nothing. There will then be abundance of motion but no force, and therefore no "activity."

To make use of the readily detachable portion of energy may not be a very simple thing, and commonly requires a machine, sometimes an ingenious machine. Give a savage a charged Leyden jar, and he will probably detach from it its available energy pretty soon. But give him a charged storage-battery and he will not know what to do. A bit of thin wire, however, is all the mechanism absolutely needed in order to afford him some light and heat; if mechanical motion is required then some form of dynamo or electromagnetic motor must be supplied. Often and often we do not know how fairly to utilise even the portion of energy automatically streaming off from a body—from a gas-flame for instance when we need light. To utilise more of a body's energy than it will part with is impossible; but the progress of science may conceivably teach us, not only how to utilise the whole of what bodies already freely give off, but also perhaps even how to make bodies (say molecules for instance) part with much energy which at present, if left to themselves, they permanently retain.

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* Radiation is not heat, but another quite distinct form of energy. The phrase "radiant heat" is responsible for immense confusion.

† Work and heat may be coaxed out of a body below the temperature of surrounding objects, as, for instance, by letting air escape from a high-pressure reservoir; but such a process is not cyclical until the air is put back again, and by that operation the heat has to be put back too. Heat by itself cannot flow uphill at all.

The first and most general portion of the second law of Thermodynamics, stated above in italics, will always remain true, even when the second part, about the non-uphill flow of heat, has by future discovery been upset; because the application of a machine for the purpose of extracting otherwise retained energy, not by a process analogous to pumping but by enabling it automatically to flow whereas without the machine it could not, can hardly be regarded as other than automatic; else would the present machines for directing the flow of already available heat be liable to a similar objection. Even if the contrivance necessary for extracting molecular energy turn out to be a live thing,—and this Dr. Johnstone Stoney* most suggestively conceives it possible may be the function of some bacteria [a remarkably appropriate demoniacal function for the producers of disease], yet life, too, so far as it falls into the scheme of physics, must be considered as an automatic process, and only the energy which by any device a body can be made automatically to yield without pumping can ever be utilised. But statements about heat not flowing up hill, or about not cooling bodies below surrounding objects, or about $\int \frac{dQ}{T} = 0$, or $dQ = Td\phi$, &c.; these are liable to ultimate modification with the progress of science, since the very terms Heat and Temperature are undynamical blinkers appropriate to the consideration of particles in the lump.

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* Phil. Mag. April 1893.

APPENDIX 1.—*The Objectivity of Energy and the Question of Gravitation.*

Mr. Heaviside[*] freely contemplates the flux of energy, but declines to admit its identity, or as he calls it objectivity. And he further doubts my proposition about transformation accompanying transference, because, he says, "convection of energy," *i.e.* simple locomotion of stored energy, "is a true flux." So it is, but it is not what I meant by *transfer*. Locomotion is so absolutely essential to translational kinetic energy that I hardly think it can be desirable ever to speak of mere locomotion as transfer, even although the moving thing be a bent bow or stretched spring. It is, however, a question of convenience, and undeniably convection must enter into a flux equation, for a bullet entering a partitioned-off region of space brings into it energy which was not there before, and, when it leaves, conveys it out again.

My proposition amounts to just this, that whatever energy appears in a bounded region must necessarily have passed through the boundary. This, if true, seems to me to confer upon energy the same kind of identity or continuous existence (or if you please objectivity) as matter possesses.

The ordinary law of conservation does not assert or contemplate continuous existence: it has no objection to seeing energy disappear from existence in one place provided an equal quantity reappears somewhere else—say inside a bounded region. Either it does not attend to or believe in the fact of transfer, or else it is satisfied with a kind of fourth-dimensional out-of-space path.

As to "objectivity" or "reality," there are always metaphysical difficulties about predicating that; and Mr. Heaviside's objection that since motion is relative, energy can hardly be absolute, must be allowed due weight. This is a point on which Professor Newcomb has written in Phil. Mag. February 1889; arguing that it actually limits the generality of the law of conservation. I hope some day to discuss this at more length; meanwhile my belief is that it will be ascertained that motion with respect to the ether is the energetic thing and that other absolute motion is meaningless.

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[*] Phil. Trans. 1892, p. 427, "On the Forces, Stresses, and Fluxes of Energy in the Electromagnetic Field," by Oliver Heaviside, F.R.S.

The fact (if fact it be) that energy has a continuous existence, or that if it appears in a closed region it must have penetrated the walls, is expressed quite clearly by Mr. Heaviside's equation

$$\text{conv}\, eu = \dot{e},$$

where e is the energy per unit volume, u is its velocity of locomotion, and where *conv.* stands for $S\nabla$, or $d/dx + d/dy + d/dz$.

But Mr. Heaviside is not satisfied with this simple equation of continuity, and proceeds to complicate it by introducing:—

(1) Intrinsic sources; *i.e.* creation or fourth-dimensional apparition of energy; of which the chief example is the gravitation bogey, whose path and nature no man yet knows.

(2) Flux of energy travelling not with matter at a definite speed, but in some other way so that its speed is uncertain. For example of this he instances radiation, but surely that has a definite enough velocity. He might have instanced conduction of heat; but there again, treated merely as a flux of energy, the amount crossing unit area per second is definite enough. Mr. Heaviside would probably agree, but would prefer not to analyse it into two factors e and u; and to this I cannot object.

(3) But to his third category Q, the rate of waste of energy, I am bound to object. The insertion of dissipation of energy as if it were a mysterious disappearance term, is open to the objection suggested above against (1), and also to the objection that it unduly elevates the available portion of energy into being the whole of it.

So long as these various terms are only introduced for practical purposes, *i.e.* to direct attention to what might otherwise get overlooked, they are well enough; but they must not be supposed to represent the reality of things. It is true that the case of gravitation, if it be transmitted instantaneously, as seems not unlikely, is a curious simulacrum of action at a distance; whereby of course energy could be generated *de novo* inside closed boundaries readily enough; but infinite speed of transmission only requires infinite incompressibility in a medium, it does not dispense with a medium; and if a medium of transmission exists, as all analogy and coherence urges if not insists, then gravitation is no exception, and its energy must pass through the walls in order to get inside a boundary, although it may pass through at an infinite pace. It may be better, however, not to assume the pace infinite till proved, but to have a term in the most general energy-equations expressive of the possible propagation of gravitation in time, notwithstanding that its speed is unknown and certainly excessively great.

I may refer to another reply I have made to Mr. Heaviside in 'Nature,' vol xlvii. p. 293 (January 26, 1893).

APPENDIX 2.—*More detailed Discussion of the Transmission of Energy in difficult cases.*

When I say (as I do on pp. 304, 305, and 311) that acting bodies have the same velocity, I of course mean their action to be immediate. If indirect action is contemplated, it is too obvious that a clock-weight has not the same speed as the tip of the second-hand or the hammer of its bell; but at the contact of every cog, two wheels, driver and driven, are moving at the same pace. But now, it may be asked, if all action is contact-action, if all action is direct, how is it possible ever to get a variation in speed? This is a question worth answering.

It is done and only done by means of rotation. The type of all such actions is a rotating wheel propelled by an uncentral force. In such a wheel, regarded as a single rigid body, we have every gradation of speed from a maximum to nothing, and we can make use of or transmit elsewhere what speed we like. This is the essence of levers, and mechanism in general: without rotation the speed of all parts is the same, and therefore the same as the point to which the driving force is applied.

But now, treating the wheel as what it is—an assemblage of particles—how comes it that they can act on each other so as to generate differences of speed? How can a force applied tangentially to the face of a sphere cause part of the opposite hemisphere to move backwards?

If we accept the sphere as a rigid body, nothing is easier than to equate the momentum generated to the impulse of the applied force, and its moment of momentum to the moment of the impulse; but if we treat it as an assemblage of connected particles it is not so easy to tackle the problem. As is well known it did historically give trouble, until it was realised, on the ground of Newton's third law (or D'Alembert's Principle as it was called), that all internal stresses balanced each other, and might therefore be ignored for the purpose of deducing the final result.

There is now no controversy as to final result; the only question is how universal contact-action, with equal velocity between agent and patient, or driver and driven, can account for the ultimate result of all grades of velocity through zero even to minus.

There is no need to take refuge behind any such blinkers as D'Alembert's Principle: an assemblage of connected particles can be directly contemplated. Let one of them receive a blow, it is passed on to the others and the momentum spreads laterally by oblique impacts, the amount of obliquity being limited only by the arrangement of the molecules; and the component of the blow or thrust transmitted in any direction is diminished in accordance with the suitable cosine law. The process is notoriously not an easy one to follow into detail, even in a simple case, partly for lack of data; and there is some uncertainty as to the disposal of the energy for the case of a blow, though even in that case there is no uncertainty about the momentum; while for a steady force the body, however essentially elastic, gets rapidly into a practically rigid state, and the molecules then merely act as transmitters of the energy. They are, as it were, connected by massless struts and ties, and along these the energy is transmitted, partitioning itself off into several directions, much as it did in the case of impact, and producing local velocities determined by the arrangement of the particles, that is, by the shape and other circumstances of the body*.

The conception of a rigid body, to which a couple can be applied, and which moves as a whole without dislocation of parts, every portion instantaneously feeling whatever force there is to reach it, simplifies problems enormously; and it may be said that just as a moving body retains its kinetic energy and carries it through space without transfer or transformation, so a rigid body conveys thrust or potential energy through space, receiving it at one point, delivering it up at other points, and transmitting it instantaneously without transfer or transformation. The thrust of a connecting-rod, the torque of a shaft, the tension of a belt, and the tangential stress of a cog-wheel, are typical instances of this practically instantaneous communication, or locomotion of potential energy, caused by a rigid body.

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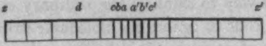
* In Thomson and Tait, vol. i. part 1, §§ 311–318, the effect of an impact on a number of elastically connected particles is treated. So it is partly in a review in 'Nature,' vol. xlvii. p. 601 (April 27, 1893).

The simplest way to think of the ordinary case of gearing and shafting is thus to ignore its molecular structure and treat it as a linkage of entirely rigid bodies, where the potential energy communicated to one point is conveyed elsewhere as a simple flux without transfer or transformation, as kinetic is conveyed when a bullet is shot across an empty space.

But that this blindfold treatment does not exhaust the matter can be seen at once by thinking of a moving fly-wheel suddenly geared on to a stationary cog-wheel, so as to transmit a portion of its kinetic energy to machinery. The stress necessary to effect the transfer is too gigantic, and results in damage unless some elasticity is provided in shaft or spokes wherewith to store it temporarily and subsequently give it out again in the kinetic form, or unless by means of slip a good deal of energy is dissipated. But some dissipation is essential anyhow: a certain fraction, equal to the ratio of the moving mass to the whole mass, must assume a vibrational form of some kind.

It is instructive to recollect certain elementary facts, such as that in the impact of two perfectly elastic bodies, one moving, one stationary, all the energy and all the momentum *may* be transmitted complete, but that in this case the resulting motion of the two bodies cannot be the same. Whereas, if by reason of elastic vibrations, or by reason of heat dissipation, it is arranged that two equal bodies shall after impact move together, then only one half the momentum and only one quarter of the energy is transmitted, the remaining half of the energy having been diverted or wasted in heat or vibration.

So it is with any system of shafting or mill-gearing to which motion has to be communicated from a revolving fly-wheel. The conservation of moment of momentum gives part of the circumstances; the refusal of ultimate mutual recoil gives the other part. And the amount of ultimate vibrational dissipation of energy is precisely the same, under these circumstances, for perfectly elastic as for perfectly inelastic bodies. The case is not unlike the opening of a stopcock between a full and empty pair of reservoirs. A certain fraction of the energy is necessarily either dissipated in heat or left as permanent vibration.



Consider the circumstance of the impact of a couple of equal elastic rods moving end-on. Conceive them in transverse strata labelled *a*, *b*, *c*, … *z* for each rod. Let one rod be stationary, and the other strike it longitudinally with velocity *v*. At the beginning of the impact *a* strikes *a′*, and the two move on in contact with the speed $\frac{1}{2}v$. *b* now strikes the mass *a a′*, and would accelerate it, but at the same instant *a′* strikes *b′*, communicating the motion to it and neutralizing the acceleration on itself; so that now the four strata are all moving with speed $\frac{1}{2}v$, while all the rest of one rod is still stationary, and all the rest of the other rod is still moving with original velocity *v*. The length of the half-speed piece about the point of contact continually increases, and behaves as a body under gradually increasing compression as it receives blow after blow on either end. At length *z* and *z′* strike, and the recoil is ready to begin. This is the middle of the impact, half the momentum has been transmitted and half the energy; but of the transmitted half energy, one-half again, or a quarter of the original only is in the kinetic form, the other quarter is in the potential or elastic-stress form.

If the rods are inelastic everything so far has happened similarly, but nothing further happens; what I have called the middle of the elastic impact is the end of the inelastic one. Kinetic energy from *w* and *y* has been transmitted to *w′* and *y′* through the intervention of the quasi-rigid compressed portion of each rod, and energy from *z* has reached *z′*.

A quarter of the whole energy has been transmitted direct as kinetic, and there is no potential because there is no recoil. One-half the original energy has been lost.

If the rods are elastic the recoil brings z to rest and flings z' on with the original velocity v; then y stops and y' flies on after z', and so on, till at the end of the impact the rods part asunder, the first one completely stationary, the second one completely moving with all the energy simply kinetic.

If the rods are partially elastic some of the potential energy is dissipated and some utilised, while if they are of unequal length or material the pulses are not timed similarly in both; the shorter one (supposed the striker) is struck dead as before, the longer one is left with a pulse in it after they have separated, and its residual potential energy then assumes the sound-vibrational form: the strata progressing jerkily for some time.

If the rod to be moved is incompressible, its pulse travels instantaneously and it all gets moved at once. A blow to such a rod transmits energy instantaneously, and all in the kinetic form, but there is nothing in mere speed to affect the amount transmitted.

By the consideration of instances we have thus been led to the induction that energy can be transmitted without obvious change of form by substances with infinite properties, *e. g.* by an incompressible solid; all molecular processes being either non-existent or being ignored; but that with ordinary matter there is always some percentage of obvious transformation, though we may apparently have all grades of it from complete to very small.

Thinking of these impact cases alone, it might appear as if I had been overhasty in saying that the whole of energy must be transformed when it is transferred. Yet observe that it has to pass *through* the intermediate condition. A row of ivory balls in contact has another thrown against one end, and from the other end one leaps off. The energy has been transmitted through the row somewhat as it is transmitted through the compressed strata of two impinging rods. Yet if the elastic connexions of every stratum are attended to, and if these be regarded as massless, I think it will be found that all the transmitted kinetic has really passed through a momentary existence as potential. The fact of necessary transformation is not so obvious when you come to look into some of these special cases; but I would refer once more to the proof given at the beginning of Part IV., which seems to me conclusive as to essential fact.

The difficulty arises because when an elastic body is struck (say a massive molecule with a massless spiral spring connexion) it begins to move a little directly the spring is the least compressed, and is moving half speed when the spring is fully compressed; but I venture to say that on any view of the identity of energy the bit of kinetic which it first attains is a bit of energy that has been transmitted through the elastic stress of the spring, and that just as the second half of the energy must admittedly exist in the spring before it can reach the mass, so the first half has already passed through the spring and has reached the mass only after transmutation, although the transformation is disguised while the transference is obvious.

I have now written enough to emphasize what I want to bring forward as the simple doctrine of energy. Some years ago* I attempted it with brevity, but failed to make it clear or to call proper attention to it. Now I have set it forth at length, with illustrative cases, as matter for discussion. It remains to try and formulate briefly and strictly the extended laws of motion appropriate to our present knowledge.

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* Especially in Phil. Mag. for October 1879, page 278 *et seq.*, for January 1881, p. 37, and June 1881, p. 530, and for June 1885, p. 482. Also in 'Elementary Mechanics' (Chambers), which was written in 1876 and revised about 1884, without change in the energy chapter so far as I remember; I am not responsible for dates on title-pages.

My attention has just been called to a "Smith's Prize" essay by Mr. R. F. Muirhead, communicated by Professor James Thomson to the Philosophical Magazine for June 1887. This essay, both in itself and in it numerous quotations and criticisms, is an instructive and useful summary or exposition of nearly everything that is foggy, confusing, and utterly unsatisfactory in the fundamental treatment of Dynamics. It is hardly too much to say broadly that the entire order of ideas in that essay is antipodal to the conceptions I am endeavouring to urge on the acceptance of Physicists.

DISCUSSION.

Prof. Greenhill said the paper was full of suggestive ideas. He could not agree with all the views expressed in the paper on the subject of Newton's 3rd law. In considering action and reaction as equal and opposite, Newton ignored the inertia of the medium, and if this be included, the forces are no longer equal.

Mr. Boys asked Prof. Greenhill if the first link of a chain pulled on the second link harder than the second pulled on the first?

Prof. S. P. Thompson thanked the author for getting rid of all square laws. He himself could not conceive of any effect being more than proportional to the cause.

Mr. Burbury, referring to Dr. Lodge's deduction of conservation of energy on slip 6 of the paper, pointed out that two bodies may act on one another with equal and opposite forces and yet be moving with unequal velocities, and that the kinetic energy of such a system is not conserved. He did not see that denial of action at a distance was necessary to the proof given by the author.

Dr. Stoney said that only provisional and temporary foundations of dynamics could be deduced in the way followed by Dr. Lodge. It was necessary to first ascertain how the study of Nature is related to Ontology, but this essential the author passed over by discarding metaphysics. When pursued in relation to the scientific study of Nature, Ontology throws a flood of light on most of the points dealt with in the paper.

Prof. Herroun considered that universal contact-action involves the conception of a homogeneous ether filling all space, and yet having ordinary matter dispersed through it; this he regarded as a metaphysical impossibility. If the medium be supposed molecular, then action at a distance was only postponed one step. He thought Mr. Heaviside was right in denying the objectivity of energy, for it had no more claim to be regarded as an entity than "life" or "death," "hardness" or "colour," or any other generalized abstraction.

Prof. Minchin said the first fundamental axiom of dynamics postulates the existence of *force* as an entity distinct from *matter*, *space*, and *time*, and this was the object of Newton's First Law. It also gave the criterion of the presence of force. To merely retain the law as defining *equal times* was to degrade it. As regards the supposed impossibility of defining uniform motion, he said similar difficulties occur in all sciences, even in geometry. Nevertheless, a rational science of geometry existed. In dynamics we had notions of a right line and of uniform motion in it, although no criterion of either may exist. The fact that the science harmonises with ordinary experience constitutes its validity. In his opinion the extraordinary devices which had been suggested for defining directions fixed in space were unnecessary, and merely served to cover the subject with ridicule.

He disagreed with Prof. Lodge in admitting the first law as a particular case of the second, for unless force was postulated (the function of the first law) the second became a mere definition, and not a law. Speaking of the third law, he said the author had made a serious error in stating that it could be deduced from the first, for the centre of mass of a system might be at rest without action and reaction necessarily being equal and opposite. The third law was not superfluous; neglecting it had led to great misconception and mystery about the principle of virtual work, and D'Alembert's principle, both of which are simple deductions from it.

In opposition to Dr. Lodge, he defended the ordinary definition of energy, and asserted that without the notion of *force* and *work*, the term *energy* loses all meaning.

Speaking of transference and transformation of energy, he inquired if the proof given could be applied to the case of a body sliding down a rough, rigid, inclined plane, for here the stress (friction) does work on the body but not on the plane, and there was no transference. He regretted that the expression "potential energy" was used in different senses in the paper, sometimes meaning "static energy," and at others "the available portion of the kinetic energy of a body."

Referring to the idea of all energy being ultimately kinetic, he asked if, by accepting this, the author meant to surrender the independent existence of force. If so, difficulties would arise; for example, in the kinetic theory of gases the expression for the pressure, $p=\frac{1}{3}\rho^{v^2}$, was only arrived at by assuming the existence of force. The statement on the top of slip 9 about making a "moving body do work" was not necessarily true, as might be seen by considering the case of a sphere rolling down a rough inclined plane.

Prof. O. Henrici thought axioms should be treated as true logical definitions, as for example in geometry. "Two straight lines cannot enclose a space." Every new notion required its axiom. In passing from geometry to kinematics, the idea of time presented itself, and the appropriate axiom was contained in Newton's first law. On approaching dynamics, force and mass were met with. He disagreed with Prof. Minchin in regarding force as most fundamental. Mass was more essential, for force might be abolished. On the other hand, he concurred with Prof. Minchin in thinking that the idea of a centre of mass was not axiomatic. Referring to Dr. Lodge's summary ('Nature,' May 18, 1893), he agreed with axiom (*a*) fully, and with (*b*) partially. Axiom 3 required further development. The critical point, however, was axiom 4: "Stress cannot exist in or across empty space." This he regarded as very incomplete, and maintained that axioms defining the properties of the ether were necessary to further progress. If varieties of space be contemplated, each advance required fresh axioms.

Dr. C. V. Burton remarked that contact movement did not necessitate equal velocities; sliding motion was a case in point. Again, in deforming an incompressible fluid, although force and motion might exist, no work was done. Conservation could not be proved from denial of action at a distance. Speaking of the doctrine of transference and transformation of energy, he said it was a convenient working rule, but not true universally. Newton's laws were simple and consistent, but some doubt existed as to how much was definition and how much *law*, or fact.

Mr. Founier d'Albe disagreed with Prof. Lodge in regarding the conception of uniform motion as a primary muscular sensation. It would be more correct to say that the conception was based on optical sensation, whilst the idea of force was derived from muscular sense. Newton's first law was not an axiom, for it could be proved experimentally.

He thought Prof. Lodge's criterion of the identity of energy quite sufficient, if taken in conjunction with its conservation. The difference between the identity of energy and of matter lay in the number of attributes by which they could be identified, energy having only one, viz., quantity.

Prof. Ayrton said the best foundations of dynamics depended on what was most easy to grasp. On this point great difference of opinion existed; some persons thought the idea of *force* more simple than those of *mass* or *time*, whilst others had contrary impressions. He could not admit that the conservation of energy could be deduced from denial of action at a distance. Experiments were necessary. In addition to the case previously mentioned of a body sliding on a plane, he thought a hard magnet acted on by a coil approaching it and conveying a current was one in which Prof. Lodge's law of transference and transformation did not hold.

Mr. Swinburne protested against difference between theory and a working hypothesis being overlooked. All conceptions were based on experience, and ideas of ether and atoms derived from "jelly" and "cricket balls." We ought also to remember what "explanation" means, viz., describing the unfamiliar in terms of the more familiar. It was customary to describe the phenomena of fluids by reference to solids, because we were more familiar with solids; an intellectual fish would probably do the reverse. The so-called "theory of magnetism" which breaks up a bar of iron into a number of small pieces, each possessing the properties of the original bar, he regarded as absurd. It was no "explanation" and not a "theory." Ether might be used as a working hypothesis, but must not be treated as an entity.

Mr. Blakesley questioned whether transference of energy was always accompanied by transformation, and he did not see why energy should be looked upon as

$$(mv) \cdot \frac{v}{2},$$

in preference to any other subdivision of the factors. As regards effects being proportional to their causes, he pointed out that the variable part of the E.M.F. at the junction of two metals varies as the square of temperature measured from a fixed point, and therefore this E.M.F. was always in the same direction.

Prof. S. P. Thompson, referring to the demonstration of the law of transference, &c., given on slip 8, said that attempts to translate it into Latin or Greek at once revealed the ambiguous character of the proof. Speaking of Ohm's law, he pointed out that R, a constant, was not an essential feature as Dr. Lodge supposed. Ohm never said R was constant.

In identifying energy, a difficulty presented itself, for one never came across it as a single thing, but as a product, and in being transformed the paths of the two factors might possibly be different.

Mr. Dickson said the whole of geometry and dynamics could be based on verbal definitions. The conservation of energy could be written as

Kinetic energy + potential energy = a constant,

but on substituting the expressions for kinetic and potential energies, an identity resulted; therefore the original statement was not a law. Both the kinetic and potential energies of a system were functions of its configuration. Potential energy could not belong to a particle but to a system.

The President doubted whether Dr. Lodge's scheme was more simple, natural, and logical than the ordinary one. The statement in 'Nature' (May 18th) that "strains were proportional to stresses" was simple enough, but it was questionable if "frequency of vibration is independent of amplitude" could be considered so. The author appeared to ignore *mass* in comparison with *force*, whereas the idea of *mass* seemed the more simple one. Referring to the identity of energy, he said that however far we trace it, we cannot identify its parts in molecular structure. He objected to carrying too far the ideas derived from the matter in mass to particles, and pointed out that by so doing the difficulty was only pushed one step further and not cleared up.

Dr. Lodge, in reply to Mr. Burbury, said two bodies never do attract one another; the thing which acted on either was the medium immediately in contact with it. Mr. Herroun had used metaphysical arguments against ether, but he (Prof.

Lodge) thought it was a good thing to investigate ether. He agreed with what Prof. Minchin said about force and the first law of motion. *Force* was the more fundamental, but *mass* was best as a standard unit. As regards ether, he was prepared to say that it has no motion. It possessed electromagnetic kinetic energy, and probably all the stress energy that exists. Referring to the slipping body mentioned by Prof. Minchin and Dr. Burton, he said that in speaking of the velocities of acting and reacting bodies being equal, he always meant that their velocities along the line of action were equal. The action between the sliding body and plane was a "catch and let go" one, like a fiddle-bow and string. On the second law of thermodynamics he hoped to say something in a subsequent paper. When he spoke of R being constant as the essence of Ohm's law, he meant constancy as regards terms which appear in the equation

$$\frac{E}{C}=R.$$

XIX. *On the Methods of Theoretical Physics.*

By Ludwig Boltzmann.

[This Article was written for and published in the 'Catalogue of the Mathematical Exhibition,' which the Association of German Mathematicians had arranged to be held in 1892 at Nuremburg, but which was at the last moment postponed to September 1893 at Munich.

It contains such a clear exposition of the views held at different times about the methods in Mathematical Physics, and more especially of Maxwell's views, and of his use of models and apparatus constructed to imitate dynamically natural phenomena, that the Council of the Physical Society have thought it desirable to make it more easily accessible to English Physicists by publishing this translation.]

Called upon by the Editors of the Catalogue to deal with this subject, I soon became aware that little which is new could be said, so much and such sterling matter having in recent times been written about it. An almost exaggerated criticism of the methods of scientific investigation is indeed a characteristic of the present day; an intensified Critique of Pure Reason we might say, if this expression were not perhaps somewhat too presumptuous. It cannot be my object again to criticise this criticism. I will only offer a few guiding remarks for those who, without being specially occupied with these questions, nevertheless take an interest in them.

In mathematics and in geometry it was at first undoubtedly the necessity for economizing labour which led from purely analytical to synthetical methods, as well as to their illustration by models. Even if this necessity appears to be a purely practical and obvious one, we here find ourselves on ground on which a whole species of modern methodological speculations have grown up, which have been expressed by Mach in the most definite and ingenious manner. He, indeed, directly maintains that the sole object of Science is economy of labour.

Seeing that in business affairs the greatest economy is desirable, it might, with equal justice, be maintained that this is simply the object of the sale-room, and of money in general, which in a certain sense would be true. Yet when we search into the distances, the motions, the magnitudes, the physical and chemical structure of the fixed stars, when microscopes are invented and we thereby discover the origins of disease, we shall not be very apt to describe this as mere economy.

But it is after all a matter of definition what we denote as an object, and what are the means for obtaining that object. It in fact depends on our own definition of existence what we recognize as existing; whether substances, or their energy, or, in general, their properties, so that we may perhaps at last define away even our own existence.

But let this pass; the necessity exists for the most complete utilization of our different powers of conception; and since it is by aid of the eye that the greatest mass of facts can be grasped simultaneously, it becomes desirable to make the results of our calculation perceptible, and that not merely by the imagination, but visible to the eye and at the same time palpable to the touch by means of gypsum and of cardboard.

How little was done in this direction in my student days! Mathematical instruments were almost wholly unknown, and physical experiments were often made in such a manner that they could only be seen by the Lecturer himself. And as, further, owing to shortness of sight I was unable to see writing on the blackboard, my imagination was constantly kept on the stretch. I had almost said to my good fortune. Yet this latter statement would be in opposition to the object of the present Catalogue, which can only be to praise the infinite equipment of models in the mathematics of the present day; and it would, moreover, be quite incorrect. For even if my powers of conception had gained, it could only have been at the expense of the range of my acquired knowledge. At that time the theory of surfaces of the second order was still the summit of geometrical knowledge, and an egg, a napkin ring, or a saddle was sufficient. What a host of shapes, singularities, and of forms growing organically out of each other, must not the geometrician of the present day impress on his memory! and how greatly is he not helped by plaster casts, models with fixed and movable strings, links, and all kinds of joints!

Not only so, but those machines make more and more way which serve not for mere illustration, but save the trouble of making actual calculations, from the four ordinary rules of arithmetic to the most complicated integrations.

As a matter of course both kinds of apparatus are most extensively used by physicists, who, in any case, are continually accustomed to the manipulation of all kinds of apparatus. Optical wave-surfaces, thermodynamical surfaces in gypsum, wave-machines of all kinds, apparatus for illustrating the laws of the refraction of light and other laws of nature, are examples of models of the first kind.

In the construction of instruments of the second kind some have even gone so far as to attempt the evaluation of the integrals of differential equations which hold equally for a phenomenon difficult to observe (like the friction of gases) and for another which allows of easy measurements (like the distribution of an electric current in a conductor of suitable shape) by observations of the latter, and then to utilize these values for the determination of the constant of friction. We may also remember the graphical utilization of the series and integrals occurring in the theory of tides, in electrodynamics by Lord Kelvin, who, in his 'Lectures on Molecular Dynamics,' suggests even the establishment of a mathematical institution for such calculations.

In theoretical physics other models are more and more coming into use which I am inclined to class as a third species, for they owe their origin to a peculiar method which is being applied more and more in this science. I believe this is due rather to practical physical needs than to speculations as to the theory of cognition. The method ha nevertheless, an eminently philosophical stamp, and we must accordingly enter afresh the field of the theory of cognition.

At the time of the French Revolution and afterwards the great mathematicians of Paris had built up a sharply defined method of theoretical physics on the basis laid by Galilei and Newton. Mechanical assumptions were made, from which a group of natural phenomena could be explained by means of mechanical principles which had attained a kind of geometrical evidence. Men were conscious that the assumptions could not with complete certainty be described as correct, yet up to a certain point it was held to be probable that they were in exact conformity with fact, and accordingly they were called hypotheses. Matter, the luminiferous æther for explaining the phenomena of light, and the two electrical fluids as sums of mathematical points were thus conceived. Between each pair of such points a force was supposed to act having its direction in the line joining the two points, and whose strength was a function of their distance, still to be determined. (*Boscovich.*)

An intellect knowing all the initial positions, and initial velocities of all these material particles, as well as all the forces, and which could integrate all the differential equations arising out of them, would be able to calculate beforehand the whole course of the universe just as the astronomer can predict a solar eclipse. (*Laplace.*)

There was no hesitation in declaring those forces, which were accepted as axiomatically given and not further explainable, to be the causes of the phenomena, and the determination of their values by aid of their differential equations to be their explanation.

To this was afterwards added the hypothesis that even in bodies at rest their particles are themselves in a state of motion, which give rise to thermal phenomena, and whose nature is especially sharply defined in the case of gases. (*Clausius.*)

The theory of gases led to surprising prognostications; thus, for instance, that the coefficient of friction was independent of the pressure, and certain relations between friction, diffusion, and conductivity for heat, &c., &c. (*Maxwell.*)

The aggregate of these methods was so successful that to explain natural phenomena was defined as the aim of natural science; and what were formerly called the descriptive natural sciences triumphed, when Darwin's hypothesis made it possible not only to describe the various living forms and phenomena, but also to explain them. Strangely enough Physics made almost exactly at the same time a turn in the opposite direction.

To Kirchhoff, more especially, it seemed doubtful whether it was justifiable to assign to Forces that prominent position to which they were raised by characterizing them as the causes of the phenomena.

Whether, with Kepler, the form of the orbit of a planet and the velocity at each point is defined, or, with Newton, the force at each point, both are really only different methods of describing the facts; and Newton's merit is only the discovery that the description of the motion of the celestial bodies is especially simple if the second differential of their coordinates in respect of time is given (Acceleration, Force). In half a page forces were defined away, and physics made a really descriptive natural science. The framework of mechanics was too firmly fixed for this change in the external aspect to have any effect on the interior. The theories of elasticity which did not involve the conception of molecules were of older date (*Stokes*, *Lamé*, *Clebsch*). Yet in the development of other branches of physics, Electrodynamics, theories of pyro- and of piezoelectricity, the view gained ground that it could not be the object of theory to penetrate the mechanism of Nature, but that, merely starting from the simplest assumptions (that certain magnitudes are linear or other elementary functions), to establish equations as elementary as possible which enable the natural phenomena to be calculated with the closest approximation; as Hertz characteristically says, only to express by bare equations the phenomena directly observed without the variegated garments with which our imagination clothes them.

Meanwhile several investigators had, from another side, assailed the old system of centres of force and forces at a distance; it might be said to have been from the exactly opposite side, because they were particularly fond of the variegated garment of mechanical representation; it might also be said to be from an adjacent side, as they also dispensed with the recognition of a mechanism lying at the basis of the phenomena, and in the mechanism which they themselves invented they did not see those of Nature, but mere images and analogies*. Several men of science, following the lead of Faraday, had established a totally different conception of Nature. While the older system had held the centres of force to be the real, and the forces themselves to be mathematical conceptions, Faraday saw distinctly the continuous working of the forces from point to point in the intermediate space. The Potential, which had hitherto been only a formula for lightening the work of calculation, was for him the really existing bond in space, the cause of the action of force itself. Faraday's ideas were far less lucid than the earlier hypotheses, defined as they were with mathematical precision, and many a mathematician of the old school had but a low opinion of Faraday's theories, without, however, by the clearness of his own conceptions making such great discoveries.

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* The relation of the old system of centres of force, and of forces at a distance to the purely mathematical one represented by Kirchhoff, and to Maxwell's own point of view is expressed by him in the following words:—"The results of this simplification may take the form of a purely mathematical formula (*Kirchhoff*), or of a physical hypothesis (*Poisson*). In the first case we entirely lose sight of the phenomena to be explained, and though we may trace out the consequences of given laws, we can never obtain more extended views of the connexions of the subject. If, on the other hand, we adopt a physical hypothesis, we see the phenomena only through a medium, and are liable to that blindness to facts and rashness in assumption which a partial explanation encourages. We must therefore discover some method of investigation which allows the mind at every step to lay hold of a clear physical conception without being committed to any theory in physical science from which that conception is borrowed, so that it is neither drawn aside by analytical subtleties, nor carried beyond the truth by a favourite hypothesis."

Compare the theory of elasticity worked out by Kirchhoff in his Lectures, of almost etherial delicacy, clear as crystal but colourless, with that given by Thomson in the third volume of his Mathematical and Physical Papers, a sturdy realistic one, not of an ideal elastic body but of steel, india-rubber or glue; or with Maxwell's language, often almost childlike in its naïveté, who, right in the middle of his formulæ, casually gives a really good method of removing grease-spots.

But soon, and especially in England, it was attempted to get as plain and tangible a representation of the ideas and conceptions which before had played a part in Analysis only. From this endeavour towards clearness arose the graphical representations of the fundamental conceptions of mechanics in Maxwell's 'Matter and Motion,' the geometrical representation of the superposition of two sine motions, and all illustrations due to the theory of quaternions, for instance the geometrical interpretation of the symbol

$$\Delta = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}.^*$$

There was another matter. The most surprising and far-reaching analogies were seen to exist between natural phenomena which were apparently quite unlike. Nature seemed in a certain sense to have built up the most diversified things after exactly the same pattern; as the analyst dryly observes, the same differential equations hold for the most diversified phenomena.

Thus the conductivity of heat, diffusion, and the propagation of electricity in conductors takes place according to the same laws. The same equation may be considered as the solution of a problem in hydrodynamics or in the theory of potential. The theory of vortices in fluids as well as those of the friction of gases exhibits the most surprising analogy with that of electromagnetism, &c., &c. Compare on this point Maxwell, 'Scientific Papers,' vol. i. p. 156.

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* Maxwell, 'Treatise on Electricity and Magnetism,' 1873, vol. i. art. 29: on the nature of the operator ∇ and ∇^2; this was also afterwards observed by others. Mach on M. Guebhard's representation of Equipotential Curves, *Wien. Sitzungsbericht*, vol. lxxxvi. p. 8, 1882. Compare also Riemann, 'Electricität und Magnetism," *Wied. Beiblätter*, vol. vii. p. 10; *Comptes Rerdus*, vol. xcv. p 479.

Maxwell also, when he undertook the mathematical treatment of Faraday's conceptions, was from the very outset impelled by their influences into a new path. Thomson [*] had already pointed out a series of analogies between problems in the theory of elasticity and those of electromagnetism. In his first paper on Electricity Maxwell [†] explained that it was not his intention to propound a theory of electricity; that is, that he himself did not believe in the reality of the incompressible fluids and of the resistances which he there assumes, but that he simply intends to give a mechanical example which shows great analogy with electrical phenomena, and he wishes to bring the latter into a form in which the understanding can readily grasp them [‡].

In his second paper he goes still further, and from liquid vortices, and friction wheels working within cells with elastic sides, he constructs a wonderful mechanism which serves as a mechanical model for electromagnetism. This mechanism was of course mocked at by those who, like Zöllner, regarded it as an hypothesis in the older sense of the word, and who thought that Maxwell ascribed to it a real existence: this he decidedly repudiates, and only modestly hopes "that by such mechanical fictions any one who understands the provisional and temporary character of this hypothesis will find himself rather helped than hindered by it in his search after the true interpretation of the phenomena." And they were so helped; for by his model Maxwell arrived at those equations whose peculiar, almost magical power Hertz, the person most of all competent to judge, thus vigorously depicts in his lecture on the relations between Light and Electricity:—"We cannot study this wonderful theory without at times feeling as if an independent life and a reason of its own dwelt in these mathematical formulæ; as if they were wiser than we were, wiser even than their discoverer; as if they gave out more than had been put into them." I should like to add to these words of Hertz only this, that Maxwell's formulæ are simple consequences from his mechanical models; and Hertz's enthusiastic praise is due in the first place, not to Maxwell's analysis, but to his acute penetration in the discovery of mechanical analogies.

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[*] Cambridge and Dublin Math. Journal, 1847; Math. and Phys. Papers, vol. i.

[†] Maxwell, on Faraday's Lines of Force. Cambridge Phil. Trans. vol. x; Scient. Papers, vol. i. p. 157.

[‡] Maxwell, Scientific Papers, vol. i. p. 157.

It is in Maxwell's third important paper * and in his Text-book that the formulæ more and more detach themselves from the model, which process was completed by Heaviside, Poynting, Rowland, Hertz, and Cohn. Maxwell still uses the mechanical analogy, or, as he says, the dynamical illustration. But he no longer pursues it into detail, but rather searches for the most general mechanical assumptions calculated to lead to phenomena which are analogous to those of Electromagnetism. Thomson was led, by an extension of the ideas which have already been cited, to the quasi elastic and the quasi yielding æther, as well as to its representation by the gyrostatic-adynamic model.

Maxwell of course applied the same treatment to other branches of theoretical physics. As mechanical analogies may be cited Maxwell's gas molecules which repel each other with a force inversely proportional to the fifth power of their distance, and at first investigators were not wanting who, not understanding Maxwell's tendency, considered his hypothesis to be improbable and absurd.

The new ideas, however, gradually found entrance into all regions. In the theory of heat I need only mention Helmholtz's celebrated memoirs on the mechanical analogies of the second law of thermodynamics. It was seen indeed that they corresponded better to the spirit of science than the old hypotheses, and were also more convenient for the investigator himself. For the old hypotheses could only be kept up as long as everything just fitted; but now a few failures of agreement did no harm, for it can be no reproach against a mere analogy if it fits rather loosely in some places. Hence the old theories, such as the elastic theory of light, the theory of gases, the schemes of chemists for the benzole rings, were now only regarded as mechanical analogies, and philosophy at last generalized Maxwell's ideas in the doctrine that cognition is on the whole nothing else than the discovery of analogies.

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* Maxwell. "A Dynamical Theory of the Electro-magnetic Field." Scientific Papers, i. pp. 526; Roy. Soc. Trans. vol. 155. p. 459, 1865.

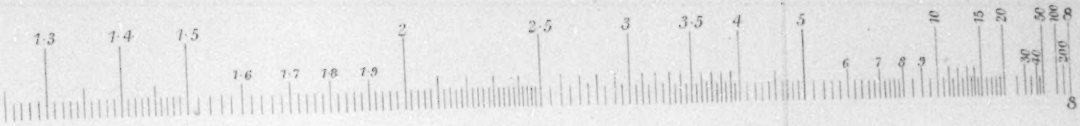

Fig. 3.

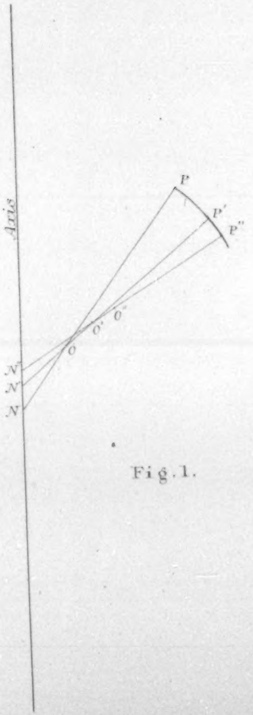

Fig. 1.

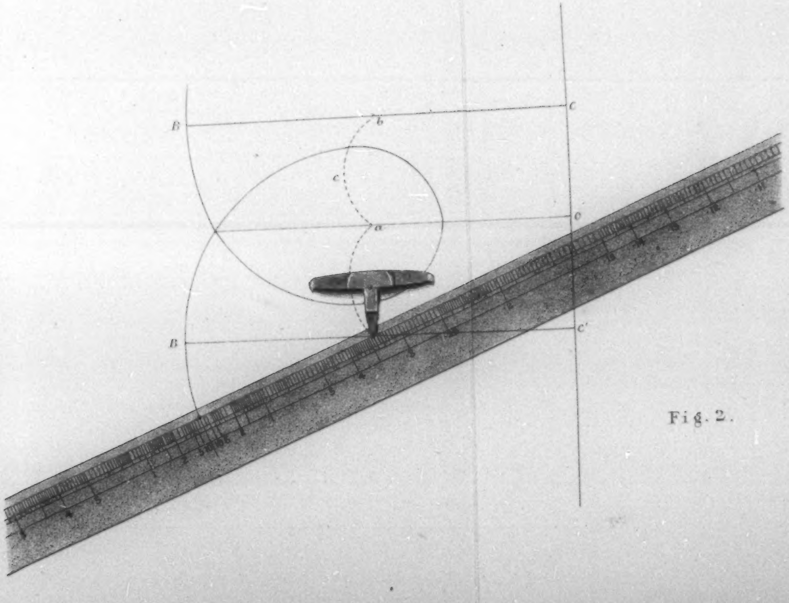

Fig. 2.

Mintern Bros. lith.

With this the older scientific method was defined out of the way, and Science now only spoke in parables.

All these mechanical models at first existed indeed only in idea; they were dynamical illustrations in the fancy, and they could not be carried out in practice, even in this general form, yet their great importance was an incitement to realize at any rate their fundamental types.

In the second part of this Catalogue is a description of such an attempt made by Maxwell himself, and of one by the author of these lines. Fitzgerald's model is also at present in the Exhibition, as well as Bjerknes' model, which owe their origin to similar tendencies. Other models which have to be classed with these have been constructed by Oliver Lodge, Lord Rayleigh, and others.

They all show how the new tendency to relinquish perfect congruence with Nature is compensated by the more striking prominence of points of similarity. To this belongs the immediate future; yet, mistaken as it was to consider the old method as the only correct one, it would be just as one-sided to consider it, after all it has accomplished, as quite played out, and not to cultivate it along with the new one.

XX. *On the Drawing of Curves by their Curvature.*

By C. V. Boys, *A.R.S.M., F.R.S.*[*]

[Plate III.]

While giving a course of lectures to working men on Capillarity at the Royal School of Mines in 1891, I wished to explain the principles upon which the form of a water-drop depends, and to show that an accurate scale-drawing could actually be produced by following Lord Kelvin's rule (Proc. R. Inst. Jan. 29, 1886); but found that the procedure was so cumbersome as to be in no way adapted to popular exposition. In my attempt to simplify the operation I devised a method of carrying it out which has the double advantage that any of these capillary surfaces of revolution can be drawn with a facility which cannot be approached by following Lord Kelvin's instructions explicitly, while the accuracy of any curve thus determined is so great that it may be used for the purpose of approximate numerical calculation when, as is often the case, the equation cannot be solved. The accuracy is certainly considerably in excess of that which can be attained by the same degree of care, using rule and compasses according to his instructions. I hope therefore that though there is no principle involved which is not perfectly well understood, the practical character of the modification and its utility more especially to teachers are such as to render it worthy of the attention of the Physical Society.

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[*] Read May 12, 1893.

Lord Kelvin's rule for drawing the generating curve of any capillary surface of revolution is as follows :—"Through any point, N, fig. 1 (Pl. III.), of the axis draw a line, NP, cutting it at any angle. With any point, O, as centre on the line NP, describe a very small circular arc through PP′, and let N′ be the point in which the line of OP′ cuts the axis. Measure NP, N′P′, and the difference of levels between P and P′. Denoting this last by δ, and taking a as a linear parameter, calculate the value of

$$\left(\frac{\delta}{a^2}+\frac{1}{\mathrm{OP}}+\frac{1}{\mathrm{NP}}-\frac{1}{\mathrm{N'P'}}\right)^{-1}.$$

Take this length on the compasses, and putting the pencil-point at P′, place the other point at O′ on the line P′N′, and with O′ as centre describe a small arc, P′P″. Continue the process according to the same rule, and the successive very small arcs so drawn will constitute a curved line, which is the generating line of the surface of revolution inclosing the liquid, according to the conditions of the special case treated."

This can be explained to those not already familiar with the principle in a few words. At any depth below the plane surface level of a liquid there is a hydrostatic pressure which is proportional to the depth. At every point of the surface of a drop this is balanced by the surface-tension which is constant over the surface multiplied by the total curvature. The total curvature at any point P in a surface of revolution is defined as being equal to $\frac{1}{\text{OP}} + \frac{1}{\text{NP}}$, where OP is the radius of curvature of the generating curve at the point P, and NP is the distance normal to the curve from P to the axis of revolution.

Since at any depth the curvature of the tense surface withstands the hydrostatic pressure and since the hydrostatic pressure is proportional to the depth, it is clear that the total curvature measured as defined above must be proportional to the depth, being concave to the liquid below and convex above the plane surface level. At any depth d the curvature c must equal $\frac{\Delta d}{t}$, where Δ is the density of the liquid (or difference of density of the two liquids if one forms a drop in the other), and t the surface-tension measured in gravitation units. Thus if Δ means grains per cubic inch, t is grains per linear inch. If a bubble is formed on a ring or between two rings, the effect of gravity is so minute that the curvature is practically the same at all levels, and the curves that are formed are the well-known roulettes of the foci of the conic sections, whose equations are known.

In order to avoid the loss of time which results from the perpetual finding of reciprocals in order to determine the new radius of curvature of each small step in the curve, I have divided a rule so that the distances of the divisions from the beginning of the scale are the reciprocals of the numbers attached to them. Such a rule is adapted to measure the smallness of a thing just as an ordinary rule measures its bigness or size. Thus a large thing since it has very little smallness is found by the rule to be measured by a very small number, while a very little thing having a considerable smallness is measured in the same way by a very large number. The curvature of a line is measured by the reciprocal or the smallness of its radius, and so the curvature of a line is read off immediately by the rule. The curvature of a surface is measured by the smallness of two lines which start at the same point and in the same or in opposite directions, and so the curvature of a surface of revolution is found by inspection of the two curvatures upon the rule. In order to be equally ready to deal with the case when the lines are measured in opposite directions, and with a further object not yet apparent, I place the zero of the scale, marked ∞, somewhere about the middle and divide it each way. Then, if the two radii are found on the same side of the ∞ or centre, their values are to be added, and if on the opposite side one is to be subtracted from the other in order immediately to find the total curvature. The only gain apparent at present is the great saving of time resulting from an obvious construction. The increase in accuracy depends upon the abolition of all accumulated errors of compass setting, which are of three kinds:—(1) The compass may not be set to the exact radius required; (2) the pencil- or pen-point may not be placed exactly upon the end of the line just drawn, producing a step; and (3) the needle-point may not be set exactly upon the former radius-line, producing an angle invisible no doubt, but still existent in the curve. The first of these applies to the method of this paper, but in less degree; the second and third are as nearly perfectly eliminated as is possible. These, though they may be individually too minute to be observed, yet tend to be cumulative, so that as the number of steps employed is increased with the view to obtain accuracy, so do these accumulated errors tend to increase when the ordinary method is followed. It is on the elimination of these that the accuracy of the method of curve-drawing, which forms the subject of this paper, mainly depends. The increase in facility is also of importance, as the operator is not tempted to make his steps unduly long; it is brought about as follows. I make the rule of a thin strip of transparent celluloid with a small hole at the centre of the scale marked ∞. A small brass tripod with three needle-feet (fig. 2) is placed so that two feet just penetrate into the paper and the third rests on the longitudinal straight line of the strip (which of course passes through the small hole), and just pricking into it forms a temporary but rigid and stable centre of rotation for the strip. A pencil-point or a capillary glass pen made short and fastened with wax or a spring clip into its place at ∞ will then draw an arc of a circle of which the curvature is equal to the reading on the strip at the place where the needle-point presses. Now it will be clear that at every stage the two numbers which represent the two parts of the curvature are always visible to the eye, there is no necessity to rule any of the normals to the curve, and no necessity to set compasses. Supposing for simplicity that a curve of constant total curvature (unduloid, nodoid, or catenoid), that is where hydrostatic pressure need not be considered, is to be drawn through any point and normal to any definite direction. Then all that has to be done is to set the strip with its longitudinal line in this direction and with its central hole at the point. *E.g.*, let the point be 6 inches from the axis, let the line through the point to which the curve is to be normal be perpendicular to the axis, and let the total curvature be arbitrarily chosen as ·5. Then when the strip is placed in position the reading on the scale where it crosses the axis will be $\cdot 1\dot{6}$. The difference between ·5 and $\cdot 1\dot{6}$ is $\cdot\dot{3}$. The needle-point must therefore be set on the scale at $\cdot\dot{3}$, and on the side of ∞ towards the axis. A very short arc must then be drawn with the pencil or pen piercing the hole at ∞. The reading on the axis will now be found very slightly less (the distance being greater), say ·16, the needle-point must now be transferred to the ·34, care being taken to hold the strip firmly upon the paper to prevent its moving. A new short arc may now be drawn and the process continued. As soon as the curve becomes notably inclined to the axis the change of reading on the axis between the two ends of a short arc even may be more than appreciable, so that if the process were carried out exactly as described the needle-point would be on the whole too late in making its move; *i. e.* where the radius is increasing it would be too small, and *vice versâ.* To avoid this it is convenient to choose such a small arc as shall change the axial reading by any small number of divisions at a time and to move the needle-point through a corresponding number at a time also, but so that the needle-point is placed always at that division which corresponds with the intermediate one of the axial divisions. The curve can in this way be carried on very rapidly by very small steps without ever even looking to see where the pencil or pen is being taken, and it in no way suffers from the small errors of each resetting of an ordinary compass. It is not even necessary to draw the curve at all. The process can be carried out as described, and the curve will be traced as in imagination by the hole at ∞. The perfection of the result which is obtainable, whether the compass or the reciprocal rule method is employed, depends of course upon the fact that whereas a very fair representation of a curve may be made by means of a polygon with a very great number of very short sides, each making very nearly an angle of two right with its neighbour, a vastly more perfect result is produced by a corresponding series of small arcs of circles, each having the necessary radius of curvature and making no angular break at each stage. The small discontinuity is one of curvature only, not of direction. The gain in smoothness and accuracy is still very great when the steps are made larger, so that a polygon of the same number of sides would cease to fairly represent the curve. Where the curvature is a maximum or a minimum the contact of the circle of curvatures is of the third instead of the second order, and at these parts a much longer step may be taken without impairing the accuracy of the curve. Similarly, if instead of the arc of curvature being used for each small step some curve of curvature and change of curvature could be employed, a still closer approximation would result, or still larger steps could be safely employed.

Going back now to the curve which has been chosen (a nodoid), it will be found that the point on the axis gets gradually farther away until the end of the rule is reached at ·05 in the case of the rule exhibited. The needle-point will now be at division ·45. For the next two or three steps the strip will be rapidly approaching parallelism with the axis, so that the axial reading is equally rapidly approaching 0; *i. e.* at an infinite distance. During these steps the axial reading cannot be directly observed; but since by the time that the rule is parallel to the axis the needle-point must have reached, in the particular case chosen, the division ·5, the two or three steps required may be made with more than sufficient accuracy by moving the needle-point a few divisions at a time so as to arrive at this point when parallelism is being reached; in fact at this stage equal angular changes correspond to equal changes on the scale. As soon as this position is passed the central line of the strip cuts the axis on the other side, and so now these readings have to be added to the total curvature ·5 in order to find the remaining readings upon which the needle-point should rest.

In fig. 2 one complete loop of this particular nodoid has been drawn, and the second one has been carried to the stage at which ·16 is the axial reading, and ·34 the reading of the temporary centre of curvature. Unduloids may be drawn in a similar way, but in this case the needle-point has to move off to an infinite distance, when the axial reading is equal to the total curvature. The length of line for which the centre of curvature is beyond the end of the rule is generally very small, and this may be drawn so that no error can be detected by alternately moving the needle from one end to the other, each for a very small arc. The catenary may of course be set out in the same way and with still greater ease and accuracy, for the infinite values do not come into any finite part of the curve. The axial reading and that on which the needle-point rests must be kept the same, but on opposite sides of the tracing-point.

It will be evident that if the little tripod is just pressed sufficiently upon the paper to make small impressions of the needle-feet, it may afterwards be taken over the same course and the third needle-point pressed upon the paper, and thus a series of points on the evolute determined. In fig. 2 the evolute is shown dotted. The points of most importance on this are the cusps *a*, *b*, corresponding to the places on the curve where the curvature is a minimum or a maximum, the distance *a b* being the half period of the curve, and the maxima or minima *c*, *c*, which are the ordinates of the curve where the abscissa is a minimum or a maximum. By means of these and the curve itself the geometrical constants of any curves drawn in this way may be determined more accurately than from the curve itself alone. Prof. Greenhill has given me the formula by which, with the use of tables of elliptic integrals, the chief constants of the unduloid and nodoid may be calculated. Miss Stevenson, a student of the College of Science, has drawn a large series of these curves, and we have found that while they can be drawn each in a few minutes only, it is possible to set them out correctly to the thickness of a fairly fine pencil-line; and so they do more than give a general idea of the shape, they serve to make a true scale-drawing.

In the case of capillary surfaces of revolution, in which the fluids on the opposite sides of the surface are of different density, *e. g.* water and air as already explained, the curvature of the surface must be zero at the plane surface level, and must be proportional to the distance above or below this level, being — or convex to the denser liquid above, and + or concave below. The readiest way to determine the curvature when drawing one of these generating curves is to work on a sheet on which a series of equidistant lines have been ruled parallel to the plane surface level, and at such a distance apart that the hydrostatic pressure due to this distance is balanced by a curvature of ·01 or some small and simple number. These lines, then, give by inspection, especially if each fifth and tenth line is a little darker, a scale of curvature, and exactly the same process that was followed when describing the curves of constant curvature may be as readily applied when the curvature is constantly varying. In the case of water this distance is, using the formula already given, $d=\frac{ct}{\Delta}$, = ·0128 for curvature $c=1$, or ·000128 for $c=\cdot01$. If, therefore, the drop or other capillary surface is drawn to ten times the true scale, to effect which the vertical scale of curvature must be multiplied 100-fold, the dimensions are such as to allow of considerable accuracy. I am able to show a very beautifully executed series of capillary curves for alcohol and water drawn to a scale of 10, which Miss Stevenson has been good enough to prepare; but as Prof. Perry has constructed a large number which are printed in the Royal Institution Proceedings, to which reference has been made, there is no occasion to publish them. He, however, would have been able to draw his in as many hours as he required weeks had he made use of the method described in this paper.

In order to originate the reciprocal rule I used a screw-cutting lathe as a dividing-engine, and scratched the wax off a strip of plate glass with a fine point at the places where divisions were required. Of course as the distance from ∞ increases and the readings become smaller, the distance between marks corresponding to a regular series of figures becomes greater, and new subdivisions are with advantage added, as is usual where a scale is not one of equal parts. The glass scale was etched, and the divisions transferred to a strip of celluloid by means of the well-known method with a radius-bar or tube to which two needles are fastened. Fig. 3 is a full-size copy of a portion of the glass scale, on which, to avoid fractions, the numbers are all multiplied by 10. The inch unit was chosen instead of the centimetre, as the leading screw had eight threads to the inch.

It is evident that any curves in which simple relations exist between the radius of curvature and the normal, the coordinates, or other easily observed function of the curve, may be readily drawn in the same way, but in general a scale of equal parts is preferable to the reciprocal scale. The gain in accuracy and perfect evenness and smoothness of the curve over what is obtainable by the use of compasses is the same as that which was found in the case of the capillary curves of revolution, but the saving in time will of course, though still considerable, be less. I am able to show as examples the catenary $\rho=-n$, the undulating and the looped elastica $\rho=\pm\frac{a^2}{y}$, the particular elastica $\rho=\frac{1}{2}n$, the cycloid $\rho=2n$ with its identical evolute, the tractrix $\rho=\frac{a^2}{n}$ with its evolute the catenary and the evolute of the catenary, the parabola $\rho=2n$ with its evolute, and all the conic sections (including the parabola) $\rho=\frac{n^3}{l^2}$.

It is, moreover, evident that the further use which Lord Kelvin has made of the method of drawing curves from their curvature in order to solve certain dynamical problems is one to which the procedure described in this paper may be applied with advantage.

I must in conclusion express my obligation to Miss Stevenson, who has assisted me in the construction of these rules, and who has spared no pains in setting out the beautifully drawn curves which I am able to exhibit.

Discussion.

Prof. Perry considered the method a new departure of great value. When he (Prof. Perry) drew the capillary surfaces of revolution in 1875, he found that cumulative errors produced considerable discrepancies.

Prof. Henrici thought the method would be a very useful one.

Prof. Greenhill said one would now be able to secure better diagrams of transcendental and other curves than heretofore, and he thought Mr. Boys's method would supplant the laborious processes now used to determine the paths of projectiles. When the resistance varied as the square of the velocity, the elevation for maximum range depended on the initial velocity, and for a cube law, both elevation and range tend to finite limits as the initial velocity increases.

Prof. Minchin inquired whether the catenary could be best drawn by using a scale of equal parts instead of one divided reciprocally.

The President greatly appreciated the saving of labour effected by Mr. Boys's method, and thought the apparatus should be shown at the forthcoming Exhibition of Mathematical Instruments in Germany.

XXI. *A New Photometer.* *By* ALEXANDER P. TROTTER [*].

IN the course of my investigations on the distribution and measurement of illumination [†], which led to the examination of the lighting of several streets and public places in London during the winter of 1891–92, I used illumination-photometers of different kinds. The final form consisted of a horizontal screen of white cardboard, or of paper mounted on glass, having a clear star-shaped hole in the middle. Below this and enclosed in a box was an inclined white screen illuminated by a small glow-lamp. By adjusting the illumination of the lower screen until the star-shaped hole more or less completely disappeared, the illumination of the horizontal screen was measured. After trying several different methods, the illumination of the lower screen was adjusted by altering its inclination to the glow-lamp. I greatly prefer this arrangement of screens to a Bunsen Photometer, first, because there is only one spot to examine instead of a pair of images, and secondly, because, under favourable conditions, an almost complete disappearance of this spot may be effected, instead of a similarity between two images, as with a Bunsen spot. A Bunsen screen with which the spot disappears on both sides simultaneously is rare. I understand that by warming the grease-spot the edge may be softened, and that simultaneous disappearance may be secured. I have never seen such a screen, and am inclined to think that the disappearance is illusory, and that such softening is probably accompanied by decreased rather than by increased precision.

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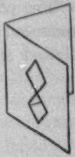
[*] Read June 9, 1893.

[†] "The Distribution and Measurement of Illumination," Proc. Inst. Civil Engineers, vol. cx. pt. iv. Paper No. 2619.

I have recently applied my arrangement of screens to ordinary light-photometry (as distinguished from illumination-photometry). My first plan consisted of two screens (fig. 1), each inclined at 45° to the direction of the lights and to the eye. One screen was immediately behind the other; the front screen was perforated, and was mounted on a sliding-carriage on a photometer-bar. The lights were placed, the one a little in front and the other a little behind the plane of intersection of the screens. The back of the perforated screen was blackened and was shaded from the light which illuminated the back screen. The edge of the perforation was bevelled, to assist the complete disappearance of the hole. The hole consisted of two lozenge-shaped apertures one over the other, point to point, the object being to concentrate attention on a vertical line. The screens were held in a frame capable of rotation round a vertical axis through a small angle, for the purpose of producing small and rapid variations. But although one screen thus received more light and the other less, the cosine-law of illumination caused the former to increase but slightly in brightness while the latter diminished considerably. It should be observed that this arrangement of screens, although developed from a Bunsen photometer, turns out to be a modification of the Thompson-Starling photometer, in which two screens at 45° to the lights* and to

Fig. 1.



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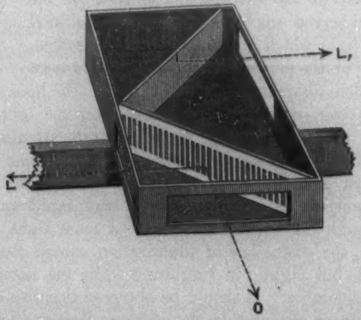
* See Note at end of the Paper.

the eye are used, but side by side, instead of one behind the other.

I showed an extemporized photometer of this kind to Mr. Crompton and to Mr. Swinburne on March 26, 1893. Mr. Crompton suggested the use of zinc for the screens, and this gave me the hint to try perforated zinc, to abandon a definite vertical line of uniform tone, and to select a line or band of uniformity on a screen provided with a number of perforations. I mentioned the idea to Mr. Swinburne on March 27, but had no opportunity of trying it for more than a week. In the meantime Mr. Crompton had thought of a similar expedient, and had used with considerable success a pair of screens, the front one of which was perforated with horizontal slots, and he observed the shading of the slots, say, from bright to dark, and of the bars from dark to bright. Before I had heard of this modification I had tried vertical bars or slots and perforated screens.

The photometer is shown in fig. 2. It consists of a box mounted on a sliding-carriage on a photometer-bar. A slotted screen and a plain screen are fixed inside, and are observed through an opening from the direction O. The lights fall on the screens through two openings from the directions L and L_1. The lights are arranged exactly opposite the middle of each side opening, and not in the

Fig. 2.



plane of the photometer-bar as is usual. The lower edge of the front opening or diaphragm carries a sight or pointer, and the back screen is provided also with a pointer, preferably at its upper edge. The photometer may be examined from a distance, say six or eight feet, and moved by cords. The two sights should be brought in a line with each other, and the carriage is to be moved until the band of uniform tone is bisected by the pointers. If the carriage be moved slowly it will be seen that the band of uniform tone remains fixed relatively to the scale on the photometer-bar. Readings might be taken by noting the position of the band over the scale, but I prefer to follow the usual custom in photometry, and to keep the scale hidden until an adjustment has been made, to avoid any bias. The operation of estimating the bisection of the band by the pointers is, I think, easier and therefore more conducive to precision than the estimation of the similarity of two images as in the Bunsen, or of two tones as in most other photometers.

I will now give details of the development of this photometer for the guidance of those who may wish to construct them.

Perforated zinc of the ordinary pattern having holes about 0·08 inch diameter and ⅛ inch pitch, although countersunk on the back and painted dead white on the front, was found to be of no use, probably because the area of the holes was considerably less than that of the remaining metal. Half-inch holes punched in thin card at 0·6 inch pitch gave on the whole a better result than ¼-inch or ⅛-inch wooden slips, bevelled, glued to a frame at a distance apart equal to their width, and painted dead white. I then tried a number of different patterns of perforated zinc and other screens, 30 centim. long by 6 centim. high. Circular perforations of 4 millim. and 6 millim. diameter, about half the metal being removed, were less satisfactory than zinc with holes 13 millim. diameter, rather more than half the metal being removed. A fancy pattern was tried, but without good results. The suitability of the screens was compared in some cases by comparison of the "mean error of a single observation" calculated in the ordinary way from ten measurements, two similar glow-lamps being used, sometimes 2 metres apart, and sometimes 3 metres.

Finding that thinness of edge and perfect flatness were very important, I made several screens of card and of paper stretched while wet on wooden frames. The coarsest of these consisted of strips of two-sheet Bristol board 8 millim. wide and 8 millim apart, 18 such strips, with an 8 millim. space at each end, making 30 centim. The strips were cut with sharp scissors to avoid any burr on the edge. Another screen has strips 5 millim. wide, and a third has 48 strips 3 millim. wide. White foolscap paper was used, soaked in hot water to remove the glaze ; it was mounted whole, and cut on glass when dry, the knife being held slanting as in mount-cutting, to bevel the edge. These screens I find, and my opinion is confirmed by friends who are for the most part unfamiliar with photometers, to be easy to use.

The advantage of vertical strips is that perfect symmetry is attained, a condition which I find to be important in photometry. When perforations are used, dark holes on a light ground are seen at one end of the screen, and light holes on a dark ground are seen at the other end. The balance is indicated by a strip at which these two patterns become confused. I have not succeeded in getting a definite easily-bisected strip of uniform tint with such screens. When, however, strips are used, the spaces being equal to the width of the strips, the appearance of the screen is perfectly symmetrical, and the effect is improved by the use of a diaphragm cutting off from the observer the edges of the screens. When lights of similar colour are compared, the light strips and dark spaces at one end are almost indistinguishable from the dark strips and light spaces at the other end. With 8 c. p. lamps 3 metres apart, and with the 8 millim. strips, one strip or one space can generally be made to disappear; that is to say, its edges become invisible when examined from a distance of 6 or 8 feet. At a shorter distance, one eye only must be used. With the 3 millim. strips, a nearly uniform grid is seen, paling a little towards the middle, and showing a band about 15 millim. wide, including, say, three strips which are indistinguishable from the two spaces between them. When lights differing in colour are used, the narrow strips are distinctly preferable : disappearance is nowhere complete, but the differently coloured strips and spaces seem to blend at a definite place on the screen.

As screens 30 centim. long and requiring a box about 50 centim. by 30 might be considered cumbersome, I have made a photometer with screens 22·5 × 3·7 centim. effective; these are contained in a box measuring about 22 × 32 × 6 centim. deep. The diaphragm facing the observer is 15 × 3·5 centim. The screens are placed at 45° to the lights and to the observer. I have tried other inclinations, but there is no difficulty in getting a surface sufficiently dead-white to work well at this angle.

I find that a distance of 3 metres between two 8 c. p. glow-lamps is rather great for easy reading of the photometer. A distance of 2 metres is perhaps a little too small; that is, if a greater distance conduces to greater precision, it might be worth while to employ an intermediate distance. Again, a distance of 3 metres seems suitable for comparing a pair of 16 c. p. lamps. This question may be considered from the point of view of a gradient or fall of illumination; and it is not unlikely that for any one observer, the "mean error of a single observation" is a function of the gradient of illumination at the photometer-screen.

With an 8 c. p. lamp at 1·5 metre, the illumination is 3·56 times a candle-metre; that is, it is 3·56 times the illumination which would be given by one candle at one metre. The gradient is 0·0474 candle-metre per centimetre. With a 2-metre photometer-bar and a pair of 8 c. p. lamps, the illumination due to one lamp is 8 candle-metres, and the gradient is 0·16 candle-metre per centimetre. With a 16 c. p. lamp on a 3-metre bar, the gradient at the middle, *i.e.* 1·5 metre, is 0·948 candle-metre per centimetre. My experience with the form of photometer described in this paper goes to show that a gradient of less than 0·1 candle-metre per centimetre is not desirable. It should be observed that the illumination on a screen inclined at 45° to the light is about 0·7 of that of a screen turned full to the light.

To graduate a photometer-bar of unit length, and having a unit light at one end, I have used the formula :—

$$l = \frac{1}{1 + \sqrt{n}},$$

where l is a length measured on the bar, and n is proportional to the light to be measured. With a 3-metre bar a displacement of 3·9 millim. from the middle is equivalent to a difference of 1 per cent. in the candle-power of the two lights. Since the band of uniformity with the 3 millim. strips may be said to be about 15 millim. wide, and 15·6 millim. are equivalent to 4 per cent., it is only by bisection of this band that a precise measurement can be made. In using the 8 millim. strips, if the middle of one strip or space is the actual point of balance, and by error the space or strip on one side of it be taken as the one indicating the balance, the error is ± 2 per cent. Out of several series of 12 observations, the mean error of a single observation has frequently fallen below this. Among some results which I consider satisfactory, compared with my experience with other photometers, are mean errors of 0·85 and of 1·48 per cent. calculated from sets of 12 readings.

Note added June 5, 1892.

In further tests with this photometer the mean errors have generally fallen below 1 per cent.

At first I tried other angles than 45°. For whitewashed metal screens this angle appears to be suitable. I find, however, that it is preferable with paper screens to use an angle of incidence of 35°, and I understand that this angle is used in the Thompson-Starling photometer.

When the light to be measured is subject to small variations, the direction and the magnitude of the variations are plainly visible by the movement of the band of uniform tone. This photometer has been chiefly used for such measurements, and seems to be well adapted for them.

XXII. *Some Notes on Photometry.*

By Silvanus P. Thompson, *D.Sc., F.R.S.*[*]

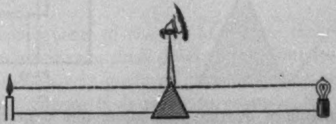
I. *On the use of Two Overlapping Screens as an Isophotal.*

The employment of two opaque screens placed at an angle with one another, and with the observer's eye, to receive the light from the two sources, was made the basis of a photometric method by the author, in conjunction with Mr. C. C. Starling, in 1881. Their photometer was designed, in the first instance, for electric-light measurements. For this special end any method in which opaque screens are used is to be preferred, in one respect, to those methods in which translucent screens are used; for then it is not necessary to employ coloured glasses when making the comparison, as the opaque screens themselves may be of coloured material, and the choice of opaque coloured material is much more varied than is that of tinted glass.

In the Thompson-Starling photometer the two surfaces for receiving the light met at an angle of about 70°. The pair of screens was constituted by two pieces of card, either white or coloured, or by two surfaces of some brilliantly-tinted fabric mounted on card, each pair being dropped down into position over a wedge-block. The observer, placing his eye opposite the *arête* of the dihedral pair of screens so as to view each surface at the same angle, had to adjust the apparatus until the apparent illumination at the adjacent parts of the two surfaces was equal.

When working with this photometer it was found that the precision of judgment of the eye as to equality of the two illuminations was impaired if by bad workmanship any considerable width of blunted edge intervened between the two

Fig. 1.



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[*] Read June 9, 1893.

surfaces that should have met with precision. There was a similar defect in the original form of the Bouguer photometer, wherein the opaque partition was continued down to the screen and interposed an unilluminated patch equal in breadth to its own thickness between the two illuminated surfaces. In that case the remedy, applied by Foucault in the modified instrument which bears his name, was to shorten the partition so as to permit the two illuminated parts to come just into optical contact. In the case of the oblique opaque screens the author tried to remedy the defect by several modifications, which, though not described at the time, proved useful. In one of these each surface was extended so as partially to overlap the adjacent surface, as indicated in figs. 2 and 3. In another instrument the overlap was given as shown in figs. 4 and 5.

Fig. 2.

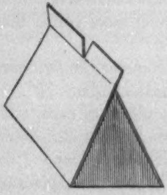


Fig. 3.

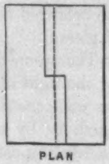


Fig. 4.

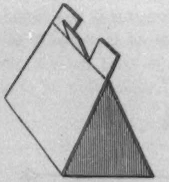
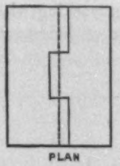


Fig. 5.



The principle of overlap in photometers operating by the diffuse illumination of opaque inclined screens has since been used by Sir John Conroy*, who employed two pieces of white writing-paper, inclined at about 60° to one another. He found, as I did, that 90° is too large a dihedral angle for exact work. Materials such as card and paper are never entirely devoid of specular reflexion. At an incidence of 45° on each surface there is so much specular reflexion of the light as quite to vitiate the observations.

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* Phil. Mag. 1883, vol. xv. p. 423.

II. *Periodic Principle in Photometry.*

It is a familiar fact to every person who is accustomed to use the Bunsen photometer that the most convenient way of taking observations is to swing the carriage which supports the screen and mirrors backwards and forwards through a continually-decreasing distance about the position of balance. That position is ultimately found by the eye noting the successive departures from equality produced by rapid small displacements on either side of the position of balance. If these inequalities can be made equal *inter se*, the zero will lie midway; for though the intensity of illumination (assuming the sources of light as points) varies inversely as the square of the distance, the change of intensity for a small change of distance will (irrespective of sign) be the same, to within a small quantity of the next higher order, whether the displacement be toward the standard light or from it. For, taking the initial distance as unity when the illumination is i, then (neglecting small quantities of higher orders), when the distance 1 is made $1+\delta r$, the proportional change of illumination is $\mp\frac{2\delta r}{1\pm 2\delta r}$; or, the change of illumination is approximately inversely proportional to the distance, and is approximately proportional directly, but with contrary sign, to the displacement.

It is a consequence of the law of fatigue that the eye is much more sensitive to small differences of illumination when these are successively produced than when they are constant in amount. Hence it is easier to estimate the true zero position on the photometer by these successive movements than by simply trying to put the carriage at zero. A difference due to a displacement of two millimetres of the stationary carriage might fail to be perceived. But the eye cannot fail to detect the difference when the carriage is quickly swung to and fro over the distance of two millimetres on either side of the zero; and, judging of the two inequalities on either side, and altering the adjustment until the two inequalities are themselves equal, the zero is determined. As a piece of pure physics, this method of successively approximating to a zero, ill-defined in itself, by means of equal errors on either side, may be compared with the method of Joule for determining the temperature of minimum volume of water.

In the researches of Abney and Festing on the photometry of colour, the same principle of estimating the position of balance by producing rapid movements giving successive inequalities on either side of the final position was used; and indeed it constitutes the basis of the method of measuring the illuminating effect of any coloured light at different parts of the spectrum. In this case also the motion was produced by hand, by shifting a lever rapidly to and fro.

Indeed it seems probable that, without any specific recognition of its meaning or importance, the method of roughly producing periodic variations in the relative intensities of the two lights has been quite commonly used by many individuals in their work with photometers.

It has seemed to the author worth while to generalize this principle, and to give definition to it, as a recognized principle in photometry. He has therefore essayed to construct a *Vibration-photometer* in which periodic changes in the illumination are deliberately produced, in order thereby to effect systematically that which has hitherto been done unsystematically and by hand.

There are several ways of arriving at the desired result:—

(1) The "screen" of the photometer may be so mounted on its carriage as itself to vibrate through a small distance, at some convenient frequency.

(2) The standard light may be moved in some periodic way to and fro through a small distance; though in the case of flame-lamps this is inadvisable.

(3) The light of the standard lamp may be made to vary by a small percentage in a periodic manner; one way being by revolving in front of it a fan with narrow

arms to obscure periodically a portion of its light; another way being to vary periodically the aperture of a Methven slit.

(4) The light that is to be measured may be made to vary periodically by some known fraction of itself in ways analogous to the preceding.

When any of these things are done the position of balance can be found by simple direct adjustment without the usual delays.

The particular form in which the author has worked out the periodic principle is by mounting upon a spring on the photometer-carriage the paraffin-block translucent "screen" of Dr. Jolly. He finds that a period of $\frac{1}{3}$ of a second is convenient for the purpose. He has also tried an eight-bladed fan, with narrow blades to obscure in passing a fraction of the light. In using with either of these periodic devices the paraffin-block apparatus of Jolly, there is found a curious optical illusion which assists the judgment. As the brightness of the illumination of the two adjacent halves of the divided block passes through the state of equality, from being brighter on the one side to being brighter on the other, the narrow dark line which divides the two luminous portions appears itself to shift, the apparent displacement being away from the more luminous side. Whether this optical effect arises from irradiation, or from any other cause, it certainly assists the eye in its judgment as to the position of balance.

Some further observations with this device are still in progress.

III. *The Electric-Arc Standard of Light.*

In 1878 experiments were made at Chatham by Abney, Cardew, and others upon the electric arc, in the course of which the practical invariability of the intrinsic illumination of the crater-surface of the positive carbon of the arc was established. Abney and Festing, in their researches on the photometry of colour, have since that period used as a standard of white light the light of the crater of the arc. As pointed out by the writer some years ago, this invariability of whiteness, which implies invariability of temperature, is necessarily due to the constancy of the temperature of volatilization of carbon. The introduction into the substance of the carbons of any material having a lower temperature of volatilization, or of any compound which has a temperature of dissociation lower than that of the volatilization of carbon, necessarily lowers the intrinsic brilliancy of the light. This having once been realized, it seemed only natural to suggest as a standard of light the light emitted from a given area of the crater-surface of a pure carbon. This suggestion was made by the author last year when writing from Rome some remarks for the discussion of the paper by Mr. Trotter on the light of the electric arc, read before the Institution of Electrical Engineers. A similar suggestion was independently made by Mr. Swinburne in the same discussion. For some time past the author has been considering the experimental methods for putting into practice the suggestion then made. The only way to secure a constant effective area of crater is to produce a much larger crater than is required, and cover it by an opaque screen pierced with a suitable aperture of standard dimensions. As this screen must be placed very near the arc, it must be kept cool artificially by circulation of water *.

As the intrinsic light of the arc is not far from 70 candles per square millimetre, a circular aperture 1 millimetre in diameter will afford a light of about 55 candles. Hence an aperture smaller than this is preferable in a photometric standard for all ordinary photometric work. For the purpose of a special photometric standard with which to compare other arc lamps, a standard light of 1000 candles or some such magnitude is doubtless advantageous. This would require about 14·3 square millimetres of crater-surface, or a circular aperture of about 4·25 millimetres in diameter. For ordinary photometric work an aperture of 0·674 millimetre diameter would give a light of about 25 candles. It is easy to ensure the illumination of the apparent aperture by employing a magnet to deflect the arc to the front face of the carbon.

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* M. A. Blondel has recently presented to the Société de Physique of Paris an apparatus denominated *arc normale,* in which is embodied this method of carrying out the proposal of Mr. Swinburne and of the author.

There is an advantage in using as a standard source of light one whose light is greater than that of the light under measurement. The light under measurement may obviously be balanced either against a smaller light brought nearer to the screen, or against a greater light placed at a greater distance. But if the greater light at the greater distance be employed, there will, for any given inequality of illumination on the screen, be required a greater actual displacement of the moving part, whether screen or light, in order to arrive at the position of balance. Hence any error in reading the scale will be a lesser fraction of the quantity to be measured.

It might have been supposed, *à priori*, that for a given total length of photometric bench (as in a Bunsen photometer from lamp to lamp) the position of the photometric screen which would make errors a minimum would be the position in the centre; balance being sought between two approximately equal lights. Geometrically it is true that a given displacement along the scale produces a minimum change in the difference of the two illuminations, if the point selected be at the centre of the scale. On the other hand, if the total length of the bench be not fixed, it is obvious that the length of bench which will, for a given change in the difference between the two illuminations, yield the greatest actual displacement along the scale will be an infinite length. But in that case, unless infinite lights are used, the two illuminations will both be zero. In fact there is another element to be considered, namely, that in order to make a satisfactory comparison, the eye requires a certain actual brilliancy of illumination in the two surfaces which are to be compared together. Doubtless the habitual patterns of photometer in use have come by natural selection to be of the size that they are in consequence of the absolute magnitudes of the standards habitually employed. If a six-foot bench is the right sort of apparatus to satisfy the needs of the eye when the standard is of one or two candle-power, obviously a longer bench will be right when standards of greater power are employed, as giving, on the average, similar absolute illuminations of the working surfaces. If the degree of accuracy implied in the possibility of reading to within 2 millimetres' length of scale can be attained with a Bunsen or Jolly photometer when the standard is a 2-candle Methven slit, or a pair of standard sperm-candles, an accuracy of five times as great ought to be attainable if there is used as a standard a 50-candle light. The bench need not in this case be five times as long as the ordinary bench: the half length only need be elongated five-fold on one side of the screen, the other half remaining as before. In other words, if the standard is of 25-fold brilliancy, it must be placed 5 times as far away from the screen as before in order to balance a given light at the same distance as before on the other side. And, so far as such standard light is concerned, a five-fold accuracy will be attained, since any error or uncertainty in reading the scale will be now but a fifth part of the whole scale-reading.

All the foregoing points, then, to the use of a brighter standard light and a longer photometric bench than heretofore. Such a standard might well be afforded by an arc crater viewed through a circular aperture 1 millimetre in diameter, giving about 0·7854 sq. millimetre of crater-surface, with a light of about 55 candles. This should be placed at one end of a bench some five metres in length; the graduations of the scale being, of course, reckoned from the edge of the aperture *.

One not unimportant advantage of the use of such a pin-hole standard is that it may with real propriety be treated as a luminous point; whereas no one can maintain that the flame standards habitually used are even approximately points relatively to the distances at which they are set from the screen. For these the law of inverse squares cannot possibly be true; though it is never the practice to make any corrections for the errors arising from the size of the flame.

In contemplating the use of circular apertures pierced in metal diaphragms, it becomes necessary to inquire how far the thickness of the metal will interfere with the illumination of the screen in directions not absolutely in line with the axis of the aperture.

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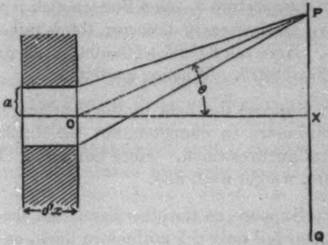
* It is curious to note in this respect that there is usually an erroneous instruction observed with the use of the Methven slit; the distance of the photometric screen being reckoned from the flame behind the slit instead of from the slit itself.

Let the screen PQ (fig. 6) be at a distance $x = OX$ from the diaphragm at O; the thickness of the diaphragm being called δx, and the radius of the circular aperture in it a. It is required to find the illumination at a point P on the screen at a distance $PX = y$ from the point X which is on the axis of the aperture. Let the angle POX be called θ. Then the apparent aperture as viewed from P will be bounded by two portions of ellipses (the front and back edges of the hole viewed in perspective), having each as semi-major axis a, and as semi-minor axis $a \cos\theta$. These two ellipses will overlap, their centres being displaced by an amount equal to $\delta x \sin\theta$. If we represent the ratio of the thickness of the metal to the radius of the aperture as $\delta x / a = \tan\phi$, then we may write the following expression for A, the effective area of the hole, as visible in the direction OP, as follows :—

Fig. 6.



$$A = 2a^2 \left\{ \cos\theta . \left[\frac{\pi}{2} - \sin^{-1}(\tan\phi . \tan\theta) \right] - \tan\phi . \tan\theta \sqrt{1 - \tan^2\phi . \tan^2\theta} \right\}.$$

Hence, since the real hole has area $= a^2\pi$, the ratio η of the illumination at P to the central illumination at X will be

$$\eta = \frac{2}{\pi} \left\{ \cos\theta . \left[\frac{\pi}{2} - \sin^{-1}(\tan\phi . \tan\theta) \right] - \tan\phi . \tan\theta \sqrt{1 - \tan^2\phi . \tan^2\theta} \right\}.$$

Case (i). To get some idea of the magnitudes involved, let us take a concrete case. Suppose the aperture to be a hole 1 millimetre in diameter and the diaphragm $\frac{1}{2}$ millimetre thick; so that $\tan\phi=1$. What will the ratio be of the oblique to the central illumination for a point P 2·5 centimetres from the centre X of a screen which is itself at a distance $OX=50$ centimetres from the aperture? As a Bunsen disk is seldom more than 5 centimetres in diameter, this is rather an extreme case. Here $\tan\theta=2{\cdot}5\div50=0{\cdot}05$. So $\theta=2^\circ\,52'$, and $\cos\theta=0{\cdot}9975$. Whence $\eta=0{\cdot}935$.

Case (ii). Suppose the hole in the diaphragm to be still 1 millimetre in diameter, but the diaphragm to be 2·5 millimetres thick. Here $\tan\phi=5$. Taking θ as before, we get $\eta=0{\cdot}684$.

Case (iii). Suppose, on the other hand, that the hole is made in metal foil only 0·1 millimetre thick, or that, having been pierced through thicker metal the edge has been cut away by countersinking, leaving only a narrow rim 0·1 millimetre in thickness. In this case $\tan\phi=0{\cdot}2$. In the case where thin foil is used there is a considerable risk of the metal buckling with heat: and it would be quite possible, from this cause alone, that the axis of the aperture should become oblique by as much as 10° from the axis of the photometer. Assuming then $\theta=10^\circ$, the ratio of the central illumination, as thus perturbed by the error of centering, to its unperturbed value will be $\eta=0{\cdot}940$.

From all this it may be concluded that unless due care be taken in the selection and proper centering of the diaphragm, errors of several per cent. may arise in the photometric measures executed with its means. The errors arising from thickness of the diaphragm diminish, whilst those arising from defect of centering are increased, as the distance from the photometer-screen to the aperture is increased. In the above investigation no account has been taken of the effects of diffraction, which in the case of very minute apertures might become important.

Discussion.

Major-General Festing, in opening the discussion on both papers, said reflection from the sides of the hole in a thick plate would tend to lessen the error calculated by Prof. Thompson. The ordinary impurities in carbon were not likely to alter the brilliancy of the crater. Capt. Abney and himself had no reason to distrust its constancy. Both the vibrating photometer and Mr. Trotter's arrangement would be very useful.

Dr. Sumpner said his photometric experience had been obtained with the Bunsen, Jolly, and Lummer-Brodhun types. With the two former the inaccuracy arising from uncertainty of adjustment was about ½ per cent.; changes of about 0·4 per cent. (average) resulted from reversing the screens. The Lummer-Brodhun instrument (which he described) was better than either of the other two, the average error being about ¼ per cent.

Mr. Frank Wright thought scientific men gave too little attention to the question of light standards. Photometers could be relied on much more than any standard at present in use. The Methven screen was the most practical standard yet devised, but in his opinion no gaseous flame could be a real standard on account of the influence of the surrounding atmosphere.

Prof. Ayrton saw difficulties in using long benches as suggested by Dr. Thompson, on account of the serious atmospheric absorption which occurs with light from arcs. Decreasing the intensity by dispersion or otherwise was preferable. In some tests on glow-lamps now being carried on at the Central Institution, a Bernstein lamp used as a standard was mounted on a spring and vibrated.

Mr. Medley showed the vibrating standard referred to by Prof. Ayrton, and gave a series of numbers showing that with this device in conjunction with the Lummer-Brodhun photometer accuracies of about ⅕ per cent. were obtainable.

Mr. Swinburne thought Mr. Trotter's arrangement was better than the "wobbling" photometer. As to the best length of bench he was inclined to think the shorter the better, provided its dimensions were large compared with those of the standard light. He concurred with Mr. Wright in his remarks about the desirability of obtaining a better standard. Speaking of the arc as a standard, he said that only impurities less volatile than carbon would influence the brightness. An important factor was the emissivity of the carbon, which might not be constant.

Mr. Blakesley thought the accuracy obtainable with Mr. Trotter's photometer had been underrated, and pointed out that by using quadrant-shaped screens intersecting orthogonally on the axis of the photometer instead of straight ones, the width of the neutral band could be greatly diminished.

Mr. Trotter, referring to Dr. Thompson's paper, said he had found considerable difficulty in making pin-holes suitable for arc standards. It was not an easy matter to accurately measure the hole when made. In photometric measurements he had found it very important to reverse his screens. Curved screens, as suggested by Mr. Blakesley, had been tried, but with little advantage. They also destroyed the approximate direct-reading property of the photometer. The subject of changing the length of a bench and its effect on the gradient of illumination was discussed. With short benches one had to guard against the departure from the inverse-square law, due to appreciable size of the standard. Recent experiments had shown that the light given out by 1 square millim. of crater-surface differed considerably from 70 candles.

XXIII. *On a new Volumenometer.*

By J. E. Myers, *B.Sc.*[*]

The instrument described below was devised by Professor Stroud some three years ago, with the view of having a volumenometer which should be capable of yielding more accurate results than could be obtained by the older methods. After the instrument was made some preliminary trials showed that considerable difficulties would be encountered in making the instrument air-tight, and in consequence of the pressure of other work the instrument was put on one side till last October, when it was placed in my hands.

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[*] Read June 23, 1893.

Various instruments, the principle of whose action is based on Boyle's law, have been devised with the object of determining the volumes of bodies without immersing them in liquids. The ordinary volumenometer devised by Say is described in textbooks, but a little calculation shows that no great accuracy can be obtained by its use. In Rüdorff's volumenometer* determinations are made by running out mercury from the cavity of the instrument, thereby increasing the volume until the pressure is reduced to a certain fixed value. After the body has been placed in the cavity of the instrument a precisely similar experiment is performed. From a knowledge of the weights of mercury which run out in the two cases, the volume of the body introduced follows by a simple calculation. In Paalzow's volumenometer† measurements are made by increasing the volume of the instrument to a certain fixed limit, first without and then with the body in the cavity. The pressure, recorded by the manometer attached, is noticed in each case, whence the volume required is calculated. Baumhauer's volumenometer‡ differs only from that due to Paalzow in a detail of construction. Messrs. Gee and Harden§ suggest a method of determining the volumes of bodies, depending on the gravimetrical estimation of the carbon dioxide occupying a vessel of known volume in which the body whose volume is required is placed. However, since a cubic centimetre of carbon-dioxide gas weighs only ·0020 gramme, it is clear that the accuracy attainable in determining small volumes by this method cannot be very great. The volumenometer about to be described has proved itself capable of determining the volumes of bodies with as great expedition and much greater accuracy than any of the instruments above noticed.

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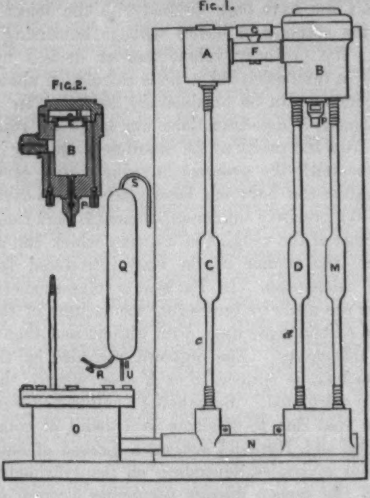
* Wied. *Ann.* vi. p. 288. † *Ibid.* xiii. p. 332.

‡ *Archives Néerlandaises.*

§ British Association Report, Newcastle, 1889.

Let A, B, C, D represent the volumes of the cavities of the vessels A, B, C, D. A is in connexion with C, and B with D, as indicated in the accompanying figure. Suppose mercury stands at the same level in each of the tubes *c* and *d*, and let the same pressure applied simultaneously compress volume C into A, and volume D into B. The condition that the mercury shall still be at the same level in both tubes is that the volume-ratios $\frac{A}{B}$ and $\frac{C}{D}$ shall be equal. This principle forms the basis of action of the instrument. The vessel B contains mercury, the quantity of which may be reduced by opening the tap P, and allowing some to run out. In this way the equality of volume-ratios may readily be obtained. The two volumes A and B can be put in communication with each other by means of the iron tube F, which is of small bore and is provided with a tap, G. The main difficulty encountered in



using each of the instruments above described arises from the necessity of hermetically closing the cavity of the instrument. This difficulty we have successfully surmounted by providing that the pressures on the under and upper surfaces of the cap employed to close the instrument shall be approximately equal. The device employed will be readily understood by reference to figs. 1 and 2. C, D, and M are glass pipettes, each of about 20 cub. centim. capacity. They differ from ordinary pipettes in one particular, the upper attachments of the bulbs consisting of glass tubes of capillary bore (1 millim. diameter). The top of the pipette C communicates with the cavity of A. By means of the screw-cap E it will be seen that the cavity of B is divided into two compartments, an upper and a lower. The pipette D communicates with the lower, and M with the upper compartment of the cavity of B. It will be clear from this, that as pressure is applied to the mercury in the reservoir it will force the mercury up the pipettes, thus compressing the air both above and below the screw-cap. An approximate balance of pressure is sufficient to guard against leakage.

The lower extremities of the pipettes are connected to an iron portion N, which is in communication with the reservoir O, containing mercury, to which the arrangement for applying pressure Q, is attached.

At the places of juncture of the pipettes and the iron portions, iron pieces whose diameter is equal to the diameter of the pipettes are screwed in. The joints are made air-tight by means of thick-walled indiarubber tubing carefully wired down. The arrangement Q employed for applying, varying, or removing pressure consists of a cylindrical tube closed at both ends, connected with the water-main by means of a narrow tube R, provided with a tap. The tube S serves to effect communication with the reservoir O, and the pressure may be reduced at any time by opening the tap on the tube U. The pipettes C and D with the connecting piece N form a convenient differential manometer.

In making determinations with this instrument, an excess of mercury is placed in the lower compartment of the vessel B. The caps E and H are screwed down to fixed marks, and pressure is then applied. Owing to the excess of mercury in B the volume-ratio $\frac{A}{C}$ will be greater than the ratio $\frac{B}{D}$. In order to equalize these ratios mercury is run out by opening the tap P. Repeated trials must be made by running out small quantities, removing and then reapplying the pressure after the vessels A and B have been put in communication with each other and with the external atmosphere. The adjustment is complete when the mercury stands at the same horizontal level in both limbs of the manometer, after application of the pressure employed. The instrument is now ready for use.

It is of importance that the pressures in the vessels A and B shall be identical at the commencement of each experiment. This equalization is effected by means of the tap G. The body whose volume is required is placed inside the cavity, the screw-cap replaced, and pressure applied to such an extent that the compression is the same as in the preliminary adjustment, as indicated by the level of mercury in the tube *c*. Owing to the diminished volume due to the introduction of the body, a further quantity of mercury must be run out in order that the manometer may not indicate any difference of pressure. The quantity which runs out is collected and carefully weighed. Calculation shows that the product of the weight multiplied by the constant of the instrument gives the volume required. For let v represent the volume of B, and let the total internal volume of B and D after the preliminary adjustment has been made $=(n+1)v$. If x is the volume of the body introduced, then the volume originally $(n+1)v$ becomes after compression $\frac{(n+1)v-x}{n+1}$. But if the volume of mercury withdrawn $=dv$, the same final volume is $v-x+dv$. Equating these expressions, we obtain the simple relation

$$x=\frac{n+1}{n}dv=k\,.\,dv,$$

where k is a constant, provided the same compression is employed in all experiments. The constant k may be readily determined by measuring with the instrument the volume of a known weight of mercury.

It is convenient therefore to start with the same initial pressure in all experiments. The initial pressure employed is that of the atmosphere, and in order to ensure that such may invariably be the case small holes have been drilled through each of the screw-caps E and H, which holes are perfectly closed by means of screws (not shown in the diagram) before the application of pressure in each experiment. The internal volumes of A and B are each approximately 16 cub. centim., and the capacities of C and D as above stated 20 cub. centim.

It is difficult to compare the instrument with those due to Paalzow and Rüdorff under similar conditions. In the present instrument air under about two atmospheres' pressure is employed, while in previous instruments rarefaction has always been resorted to. This alone suffices to produce increased accuracy. Paalzow shows, in the paper above referred to, that with his instrument a difference of pressure of 0·1 millim. involves a change of volume =0·016 cub. centim. In Professor Stroud's instrument the same pressure-difference requires a change of volume =0·0023 cub. centim.,—that is, the accuracy is 7 times as great. Rüdorff does not give the dimensions of his instrument, so that comparison cannot be made in this manner. From a series of results which he gives, however, the calculated mean error =0·008 cub. centim., and this is much larger than the error deduced from any of the series of results given below.

The following are specimens of some of the earlier results which have been obtained. At Prof. Stroud's suggestion the first uncertain figure (or estimation figure) in the result is indicated in small type:—

I. Determination of the Volume of an Iron Cylinder.

$v=3{\cdot}97_6$.
$3{\cdot}97_1$.
$3{\cdot}97_8$. Volume deduced from measurement $=3{\cdot}97_7$.
$3{\cdot}96_8$.

II. Determination of Specific Gravity of $CuSO_4\ 5H_2O$.

Weight of $CuSO_4$ employed was about 6 grammes.

The following are the results of successive determinations:—

S.G. $= 2{\cdot}28_0$. Volume $= 2{\cdot}64_0$ cub. centim.
$2{\cdot}28_8$. $2{\cdot}63_0$.
$2{\cdot}28_4$. $2{\cdot}63_5$.
$2{\cdot}28_5$. $2{\cdot}63_3$. Mean error $=$ ·003 cub.
$2{\cdot}28_6$. $2{\cdot}63_2$. centim.
$2{\cdot}28_4$. $2{\cdot}63_5$.
$2{\cdot}28_0$. $2{\cdot}64_0$.

III. Determination of Specific Gravity of a piece of Cork. The volume of cork employed was about 10 cub. centim.

S.G. $= {\cdot}172_1$. Volume $= 10{\cdot}17_9$ cub. centim.
${\cdot}172_2$. $10{\cdot}17_3$.
${\cdot}172_1$. $10{\cdot}17_9$. Mean error $=$ ·002 cub.
${\cdot}172_3$. $10{\cdot}17_1$. centim.

IV. Determination of Specific Gravity of very finely divided Cork Dust.

A value greater than the one above given for cork was of course expected. The cork dust was contained in a small cylindrical glass vessel of about 3 cub. centim. capacity and almost filled it. The experiments prove, however, that the volume actually occupied by the cork was only about 0·569 cub. centim.

S.G. $= {\cdot}95_6$. Volume $= {\cdot}56_9$ cub. centim.
${\cdot}95_4$. ${\cdot}57_1$.
${\cdot}94_9$. ${\cdot}57_4$. Mean error $=$ ·002 cub.
${\cdot}96_2$. ${\cdot}56_6$. centim.
${\cdot}95_9$. ${\cdot}56_8$.
${\cdot}95_4$. ${\cdot}57_1$.

A slight difficulty occurs when one is letting out mercury from the vessel by opening the tap, in consequence of the spasmodic change in level of the mercury in the capillary tube. A more accurate method of procedure consists in having the capillary tube graduated, and observing as follows :—Read the position of the mercury column when it is just above or below the standard position, then open the tap to permit a small drop of mercury to escape, which is separately collected and weighed, so that, after reading the position of the mercury column, a simple linear interpolation suffices to calculate the exact quantity of mercury which would have escaped if the equilibrium had been exact.

The following are successive determinations of the volume of an iron cylinder, employing this mode of procedure:—

Volume $= 6{\cdot}50_7$ cub. centim.
$6{\cdot}50_8$
$6{\cdot}50_9$.
$6{\cdot}50_8$.
$6{\cdot}51_0$.
$6{\cdot}50_9$.
$6{\cdot}50_7$.
$6{\cdot}50_9$.
$6{\cdot}50_8$.
$6{\cdot}51_0$.
Mean $= 6{\cdot}50_{85}$.

Mean error $= {\cdot}0009$ cub. centim.

Volume of cylinder deduced from measurement $= 6{\cdot}50_9$ cub. centim.

The instrument would probably be very suitable for the accurate determination of the specific gravity of samples of gunpowder.

XXIV. *The Magnetic Field close to the Surface of a Wire conveying an Electrical Current. By* Professor G. M. Minchin, *M.A.*[*]

At a meeting of the Physical Society on March 10, 1893[†], I gave an expression for the conical ("solid") angle subtended at any point, P, in space by a circle occupying any position.

When the circle is the seat of an electric current, this conical angle is the measure of the magnetic potential produced at P by the current. The conical angle is usually expressed by a series of spherical harmonics, or, rather, by two such series, one of which is to be used when the distance of P from the centre of the circle is less than the radius of the circle, and the other when it is greater. The expression which I have given consists of two complete elliptic integrals of the third kind, which for convenience I reproduce here.

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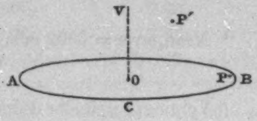
[*] Read June 9, 1893.

[†] *Ante*, p. 204.

In fig. 1 let A C B be the circular current; P′ any point in space at which the value of the conical angle subtended by the current is Ω; O V the central axis perpendicular to the plane of the current; A B the diameter of the current determined by the plane P′ O V; ν the sine of the angle P′ O V; $\rho = \mathrm{A\,P'}$, $\rho' = \mathrm{B\,P'}$; z the perpendicular from P′ on A B. Then, a being the radius O B, and $\mathrm{O\,P'} = r$, we have

Fig. 1.



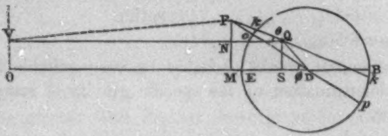


Fig. 2.

$$\Omega = 2\pi - \frac{2z}{\rho r}\left\{\frac{r-a}{1+\nu}\,\Pi\left(\frac{-2\nu}{1+\nu}, k\right) + \frac{r+a}{1-\nu}\,\Pi\left(\frac{2\nu}{1-\nu}, k\right)\right\}, \quad (1)$$

where $k^2 = 1 - \frac{\rho'^2}{\rho^2}$, and Π is the symbol for the complete elliptic integral of the third kind, its parameter and modulus, respectively, being the quantities included in the brackets following Π, according to the ordinary notation of such integrals, viz.,

$$\Pi(n, k) = \int_0^{\pi/2} \frac{d\omega}{(1 + n\sin^2\omega)\sqrt{1 - k^2\sin^2\omega}}.$$

If P′ lies anywhere in the plane of the circle and *inside* its area, $\Omega = 2\pi$; and if P′ lies anywhere in the plane of the circle and *outside* its area, $\Omega = 0$. If P′ is taken strictly *on* the circumference of the circle, and if the circle is a strictly Euclidian curve, *i. e.* something absolutely devoid of breadth,

Ω is necessarily indeterminate. It is well known that all the surfaces of constant conical angle subtended by a circuit of any form, plane or tortuous, contain the circuit as a bounding edge, and that any two surfaces for which $\Omega = \Omega_1$ and $\Omega = \Omega_2$ are inclined to each other at the constant angle whose circular measure is $\frac{1}{2}(\Omega_1 - \Omega_2)$ at all points on this common bounding edge.

When the circuit is not a Euclidian curve, but a wire, and the point in space which we consider is near the surface of the wire, as at P in fig. 1, it becomes necessary to take account of the dimension of the cross-section of the wire, and the conical angle subtended at P is the integrated result of dividing the normal cross-section made by a plane through P into an indefinitely great number of indefinitely small elements of area and breaking up the wire into a corresponding number of circular filaments having these elements of area for cross-sections, these circular filaments having all the same central axis, OV, and their planes being, of course, all parallel.

Our object now is to find the value of Ω when P is very close to the wire both when the current is assumed to be of constant density and when it is assumed to be of variable density in the cross-section. Fig. 2 represents the normal cross-section of the wire (supposed to be a circle) made by a plane through P.

Take any point, Q, in the cross-section, and at Q take an indefinitely small element of area, dS. If i is the total current flowing through the cross-section of the wire, the current in the first case through dS is $i\,\frac{dS}{\pi c^2}$, where c is the radius of the cross-section; and if Ω is the conical angle subtended at P by the circular filament of the wire passing through dS, the magnetic potential at P due to the current in the filament is

$$i\Omega\,\frac{dS}{\pi c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let the radius, QV, of the filament be denoted by a; let $PQ = R$; $\angle PQV = \theta$; then in the typical formula (1) we have $\rho' = R$, $r = VP$, $z = PN = R\sin\theta$, R being, of course, very small compared with a.

Although our object is to obtain the value of Ω correctly to the *second* order of the small quantity $\frac{R}{a}$, it will be useful for future reference to express the quantities ν, &c. as far as the third order.

Thus we have

$$r = a\left(1 - \frac{R}{a}\cos\theta + \frac{R^2}{2a^2}\sin^2\theta + \frac{R^3}{2a^3}\cos\theta\sin^2\theta\right), \quad . \quad . \quad (3)$$

$$\frac{1}{\rho} = \frac{1}{2a}\left\{1 + \frac{R}{2a}\cos\theta + \frac{R^2}{8a^2}(3\cos^2\theta - 1) + \frac{R^3}{16a^3}(5\cos^3\theta - 3\cos\theta)\right\},$$

$$\frac{1}{r} = \frac{1}{a}\left\{1 + \frac{R}{a}\cos\theta + \frac{R^2}{2a^2}(3\cos^2\theta - 1) + \frac{R^3}{2a^3}(5\cos^3\theta - 3\cos\theta)\right\}, \quad (5$$

$$\nu = \frac{R\sin\theta}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$k' = \frac{R}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where k' is the modulus complementary to k.

Now, of the two elliptic integrals in (1) the first is one in which the parameter is negative and nearly equal to -1, while in the second the parameter is positive and very large.

Observe that $\frac{z}{r} = \sin PVQ = \sqrt{1-\nu^2}$, so that (1) can be written

$$\Omega = 2\pi - \frac{2}{\rho}\left\{(r-a)\sqrt{\frac{1-\nu}{1+\nu}}\,\Pi\left(\frac{-2\nu}{1+\nu}, k\right)\right.$$

$$\left. + (r+a)\sqrt{\frac{1+\nu}{1-\nu}}\,\Pi\left(\frac{2\nu}{1-\nu}, k\right)\right\}. \quad (8)$$

To deal with the first of these elliptic intervals, let

$$\frac{2\nu}{1+\nu} = 1 - k'^2\sin^2\psi, \quad \therefore \sqrt{\frac{1-\nu}{1+\nu}} = k'\sin\psi\,; \quad . \quad (9)$$

and for the second, let

$$\frac{2\nu}{1-\nu} = \cot^2\chi, \qquad \therefore \sqrt{\frac{1+\nu}{1-\nu}} = \frac{1}{\sin\chi}. \quad . \quad . \quad (10)$$

Hence we have

$$\Omega=2\pi-2\left\{\frac{r-\alpha}{\rho}\,.\,k'\sin\psi\,\Pi(-1+k'^2\sin^2\psi,k)\right.$$
$$\left.+\frac{r+\alpha}{\rho}\cdot\frac{1}{\sin\chi}\,\Pi\,(\cot^2\chi,k)\right\}.\quad .\quad (11)$$

The values of these complete elliptic integrals of the third kind are well known in terms of integrals of the first and second kinds. Thus (see Hymers's 'Integral Calculus,' section ix., or any treatise on Elliptic Functions) we have the equation

$$\frac{k'^2\sin\psi\cos\psi}{\Delta(k',\psi)}\{\Pi(-1+k'^2\sin^2\psi,k)-\mathrm{K}\}$$
$$=\frac{\pi}{2}-\mathrm{K}\,.\,\mathrm{E}(k',\psi)+(\mathrm{K}-\mathrm{E})\,.\,\mathrm{K}(k',\psi).\quad .\quad (12)$$

Now

$$\mathrm{E}(k',\psi)=\mathrm{E}'-\int_{\psi}^{\pi/2}\sqrt{1-k'^2\sin^2\psi}\,.\,d\psi,$$

and

$$\mathrm{K}(k',\psi)=\mathrm{K}'-\int_{\psi}^{\pi/2}\frac{d\psi}{\sqrt{1-k'^2\sin^2\psi}},$$

where E′ and K′ are the *complete* integrals with modulus k'. Also it is well known that $\mathrm{KE}'+\mathrm{K}'\mathrm{E}-\mathrm{KK}'-\frac{\pi}{2}=0$. Hence the right-hand side of (12) becomes, by expanding and neglecting powers of k' beyond the second,

$$\mathrm{E}\left(\frac{\pi}{2}-\psi\right)-\frac{k'^2}{4}\left(\frac{\pi}{2}-\psi+\sin\psi\cos\psi\right)(2\mathrm{K}-\mathrm{E})\,;$$

so that

$$k'^2\sin\psi\,\Pi\,(-1+k'^2\sin^2\psi,k)$$

$$=k'^2\sin\psi\,.\,\mathrm{K}+\mathrm{E}\frac{\frac{\pi}{2}-\psi}{\cos\psi}-\frac{k'^2}{4}\left\{(2\mathrm{K}-\mathrm{E})\sin\psi+\frac{\frac{\pi}{2}-\psi}{\cos\psi}\,(2\mathrm{K}-\mathrm{E}\cos2\psi)\right\}$$

$$=\mathrm{E}\frac{\frac{\pi}{2}-\psi}{\cos\psi}+\frac{k'^2}{4}\left\{(2\mathrm{K}+\mathrm{E})\sin\psi-\frac{\frac{\pi}{2}-\psi}{\cos\psi}\,(2\mathrm{K}-\mathrm{E}\cos2\psi)\right\}.\quad (13)$$

Now since $k'=\frac{R}{\rho}$, the first term in brackets in (11) can be written

$$\frac{r-\alpha}{R}\cdot k'^2\sin\psi\,\Pi(-1+k'^2\sin^2\psi,\,k),$$

and we have

$$\Omega=2\pi-2\left\{\frac{r-\alpha}{R}\cdot k'^2\sin\psi\,\Pi(-1+k'^2\sin^2\psi,\,k)\right.$$
$$\left.+\frac{r+\alpha}{\rho}\cdot\frac{1}{\sin\chi}\Pi(\cot^2\chi,\,k)\right\}, \quad . \quad (14)$$

in which the value of the first elliptic integral is to be substituted from (13).

Dealing now with the second elliptic integral, its value is given by the known equation

$$\frac{\Delta(k',\chi)}{\sin\chi\cos\chi}\Pi(\cot^2\chi,\,k)=\frac{\pi}{2}+\mathrm{K}\{\tan\chi\Delta(k',\chi)-\mathrm{E}(k',\chi)\}$$
$$+(\mathrm{K}-\mathrm{E})\,.\,\mathrm{K}(k',\chi). \quad . \quad (15)$$

But observe that

$$\sin\chi=\frac{R}{2\alpha}\sin\theta\left(1+\frac{R}{\alpha}\cos\theta\right),$$

so that χ is a small quantity of the same order as k'. Hence, if in the coefficient of K we neglect quantities of the order $\frac{R^3}{\alpha^3}$,

$$\frac{1}{\sin\chi}\Pi(\cot^2\chi,\,k)$$
$$=\cos\chi\left\{\frac{\pi}{2}+(\mathrm{K}-\mathrm{E})\,.\,\chi\right\}$$
$$=\left(1-\frac{R^2}{8\alpha^2}\sin^2\theta\right)\left[\frac{\pi}{2}+\frac{R}{2\alpha}\sin\theta\left(1+\frac{R}{\alpha}\cos\theta\right)(\mathrm{K}-\mathrm{E})\right]. \quad (16)$$

The coefficient $\frac{r+\alpha}{\rho}$ to the second order is $1+\frac{R^2}{8\alpha^2}\sin^2\theta$, so that we have

$$\frac{r+\alpha}{\rho}\cdot\frac{1}{\sin\chi}\Pi(\cot^2\chi,\,k)=\frac{\pi}{2}+\frac{R}{2\alpha}\sin\theta\left(1+\frac{R}{\alpha}\cos\theta\right)(\mathrm{K}-\mathrm{E}). \quad (17)$$

Hence

$$\Omega=2\pi-2\left\{\frac{r-\alpha}{R}\left[\mathrm{E}\frac{\frac{\pi}{2}-\psi}{\cos\psi}+\frac{k'^2}{4}\left\{(2\mathrm{K}+\mathrm{E})\sin\psi\right.\right.\right.$$
$$\left.\left.\left.-\frac{\frac{\pi}{2}-\psi}{\cos\psi}(2\mathrm{K}-\mathrm{E}\cos2\psi)\right\}\right]+\frac{\pi}{2}+\frac{R}{2\alpha}\sin\theta\left(1+\frac{R}{\alpha}\cos\theta\right)(\mathrm{K}-\mathrm{E})\right\}. \quad (18)$$

In the small term we can put $k'^2=\frac{R^2}{4\alpha^2}$, $\frac{r-\alpha}{R}=-\cos\theta$, and $\psi=\theta$, so that, neglecting quantities of the order $\frac{R^3}{\alpha^3}$,

$$\Omega=2\pi-2\left\{\frac{r-\alpha}{R}E\frac{\frac{\pi}{2}-\psi}{\cos\psi}+\frac{\pi}{2}+\frac{R}{2\alpha}(K-E)\sin\theta\right.$$
$$\left.+\frac{R^2}{16\alpha^2}\left[3(2K-3E)\sin\theta\cos\theta+\left(\frac{\pi}{2}-\theta\right)(2K-E\cos 2\theta)\right]\right\}. \quad (19)$$

The relation between ψ and θ is given by the equation

$$k'\sin\psi=\sqrt{\frac{1-\nu}{1+\nu}}=\tan\frac{\lambda}{2},\ \text{where}\ \lambda=\angle PVQ,$$

$$\therefore\ \sin\psi=\frac{\rho}{R}\tan\frac{\lambda}{2},$$

$$\therefore\ \sin\psi=\sin\theta\left\{1+\frac{R}{2\alpha}\cos\theta+\frac{R^2}{8\alpha^2}(5\cos^2\theta-1)\right\}, \quad . \quad (20)$$

as far as the second order of small quantities; and the series for $\sin^{-1}x$ in terms of x, or rather for x in terms of $\sin^{-1}x$, gives, to the same order,

$$\psi=\theta+\frac{R}{2\alpha}\sin\theta+\frac{R^2}{2\alpha^2}\sin\theta\cos\theta, \quad . \quad . \quad . \quad (21)$$

$$\therefore\ \cos\psi=\cos\theta-\frac{R}{2\alpha}\sin^2\theta-\frac{5R^2}{8\alpha^2}\cos\theta\sin^2\theta. \quad . \quad (22)$$

Also

$$\frac{r-\alpha}{R}=-\cos\theta+\frac{R}{2\alpha}\sin^2\theta+\frac{R^2}{2\alpha^2}\cos\theta\sin^2\theta; \quad (23)$$

so that (19) becomes expressed entirely in terms of θ. Thus, $\frac{r-\alpha}{R}=-\left(\cos\psi+\frac{R^2}{8\alpha^2}\cos\theta\sin^2\theta\right)$, therefore

$$\frac{r-\alpha}{R}\cdot\frac{1}{\cos\psi}=-\left(1+\frac{R^2}{8\alpha^2}\sin^2\theta\right),$$

since θ may be put for ψ in the term of the second order; and thus we have now the equation

$$\Omega=2\pi-2\left\{(1-E)\frac{\pi}{2}+E\theta+\frac{R}{2\alpha}K\sin\theta\right.$$
$$\left.+\frac{R^2}{16\alpha^2}\left[(2K-E)\left(\frac{\pi}{2}-\theta\right)+(6K-E)\sin\theta\cos\theta\right]\right\}, \quad . \quad (24)$$

which is, however, not yet in its simplest form.

Now it is a known result that when k' is small,

$$K = \log\frac{4}{k'} + \frac{k'^2}{4}\left(\log\frac{4}{k'} - 1\right), \quad . \quad . \quad . \quad (25)$$

$$E = 1 + \frac{k'^2}{2}\left(\log\frac{4}{k'} - \tfrac{1}{2}\right), \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and these enable us to verify the above value of Ω. Thus for any point, P, in the plane of the circle and within its circumference, $\Omega = 2\pi$; and if in (24) we put $\theta = 0$, we get

$$\begin{aligned}\Omega &= 2\pi - (1 - E)\pi - \frac{R^2}{8\alpha^2}\left(2K - E\right)\frac{\pi}{2} \\ &= 2\pi,\end{aligned}$$

as we see from (25) and (26), since, to the second order, $k'^2 = \frac{R^2}{4\alpha^2}$. Similarly for any point, P, in the plane of the circle and outside its circumference, $\Omega = 0$; and this we find to be the case by putting $\theta = \pi$.

To find the value of Ω in its final form in terms of R and θ, we must substitute the value of k' in K and E. Now

$$k' = \frac{R}{\rho} = \frac{R}{2\alpha}\left\{1 + \frac{R}{2\alpha}\cos\theta + \frac{R^2}{8\alpha^2}(3\cos^2\theta - 1)\right\}, \quad . \quad (27)$$

and if we denote $\log\frac{8a}{R}$ by L, we have

$$K = L - \frac{R}{2\alpha}\cos\theta + \frac{R^2}{16\alpha^2}(L + 1 - 4\cos^2\theta), \quad . \quad (28)$$

$$E = 1 + \frac{R^2}{8\alpha^2}(L - \tfrac{1}{2}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Now in (24) K occurs only in terms of the first and second order, and therefore its value need be taken to the first order only, *i. e.*

$$K = L - \frac{R}{2\alpha}\cos\theta; \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and hence we have finally

$$\Omega = 2\pi - 2\left\{\theta + \frac{LR}{2\alpha}\sin\theta + \frac{R^2}{16^2}(6L - 5)\sin\theta\cos\theta\right\}, \quad (31)$$

which is the expression for the conical angle correct to the second order of small quantities.

As a test of the correctness of this value of Ω we should find that $\nabla^2\Omega=0$, as far as quantities of the second order. To apply this test, express Ω in terms of the columnar co-ordinates of P. Let the distance of P from the central axis OV (fig. 2) be ζ. Then we should have

$$\frac{d^2\Omega}{dz^2}+\frac{d^2\Omega}{d\zeta^2}+\frac{1}{\zeta}\frac{d\Omega}{d\zeta}=0,$$

or if $\zeta=\alpha-\eta$, where $\eta=\text{QN}$,

$$\frac{d^2\Omega}{dz^2}+\frac{d^2\Omega}{d\eta^2}-\frac{1}{\alpha-\eta}\frac{d\Omega}{d\eta}=0.$$

Now the term in brackets in (31) is

$$\tan^{-1}\frac{z}{\eta}+\frac{\text{L}z}{2\alpha}+\frac{z\eta}{16\alpha^2}(6\text{L}-5),$$

and, observing that $\text{R}^2=z^2+\eta^2$,

$$\frac{d\text{L}}{dz}=-\frac{z}{\text{R}^2},\qquad \frac{d\text{L}}{d\eta}=-\frac{\eta}{\text{R}^2},\qquad \frac{1}{\alpha-\eta}=\frac{1}{\alpha}+\frac{\text{R}}{\alpha^2}\cos\theta,$$

we find $\nabla^2\Omega=0$.

Before obtaining the effect at P due to the whole current flowing through the cross-section, $pk'\text{B}k$, it is necessary to express α, the radius of the circular filament through Q, in terms of a the radius of the filament through D. If $\text{PD}=m$, we have $\alpha=a+\text{R}\cos\theta-m\cos\phi$, where $\phi=\angle\text{PDO}$; and to the same order of approximation as before, (31) becomes

$$\Omega=2\pi-2\left\{\theta+\frac{\text{LR}}{2a}\sin\theta\right.$$
$$\left.+\frac{\text{R}\sin\theta}{16a^2}\left[(3-2\text{L})\text{R}\cos\theta+8(\text{L}-1)m\cos\phi\right]\right\}, \quad . \quad (32)$$

where $\text{L}\equiv\log_e\dfrac{8a}{\text{R}}$.

Resultant Conical Angle.—The element of area of the cross-section at Q being $d\text{S}$, the resultant conical angle subtended by the circuit at P is $\int\Omega d\text{S}$, *if the density of the current in the cross-section is assumed to be constant.* This we shall assume for the present. Let $\angle\text{QPD}=\chi$, and take $d\text{S}=\text{R}d\text{R}d\chi$. Then the first two terms in (32) will give $2(\pi-\theta)d\text{S}$, or $2(\pi-\phi+\chi)d\text{S}$, the integral of which is

$2(\pi-\phi)\,.\,\mathrm{A}$, where A is the area of the cross-section, $pk'\mathrm{B}k$. The term $\int\chi d\mathrm{S}$ obviously vanishes, since χ is negative for points Q on the lower side of the line PD.

If the tangent from P to the circle $pk'\mathrm{B}k$ makes the angle ω with PD, the values of χ run from $-\omega$ to ω; and it is obvious any integral of the form

$$\int_{-\omega}^{\omega} f(\cos\chi)\sin\chi\,d\chi$$

vanishes, f being any rational symbol of functionality, while

$$\int_{-\omega}^{\omega} f(\cos\chi)\,.\,d\chi=2\int_{0}^{\omega} f(\cos\chi)d\chi.$$

Now, the values of χ being understood to range from 0 to ω, and θ being equal to $\phi-\chi$,

$$\begin{aligned}\int\Omega d\mathrm{S}=2(\pi-\phi)\mathrm{A}-\frac{2\sin\phi}{a}\iint\mathrm{LR}^2d\mathrm{R}\,.\cos\chi\,d\chi\\ -\frac{\sin 2\phi}{4a^2}\Big[\frac{3}{2}\iint\mathrm{R}^3d\mathrm{R}\,.\cos 2\chi\,d\chi-\iint\mathrm{LR}^3d\mathrm{R}\,.\cos 2\chi\,d\chi\\ +4m\iint\mathrm{LR}^2d\mathrm{R}\,.\cos\chi\,d\chi-4m\iint\mathrm{R}^2d\mathrm{R}\,.\cos\chi\,d\chi\Big].\quad(33)\end{aligned}$$

If we first perform the integration in each case with respect to R, taking χ constant, we shall have, for example,

$$\iint\mathrm{R}^3d\mathrm{R}\,.\cos 2\chi\,d\chi=\tfrac{1}{4}\int(\mathrm{R}'^4-\mathrm{R}^4)\cos 2\chi\,d\chi,\quad .\quad(34)$$

where $\mathrm{R}'=\mathrm{P}k'$ and $\mathrm{R}=\mathrm{P}k$ (fig. 2); but since in integrating throughout the semicircle $pk'\mathrm{B}k\mathrm{O}$ the values of χ are repeated in the revolution of the radius-vector, $\mathrm{P}k'$, the values of $d\chi$ being negative as the radius-vector revolves from the position of the tangent from P, it is clear that both terms in the integral at right side of (34) are included in the single expression

$$\tfrac{1}{4}\int\mathrm{R}^4\cos 2\chi\,d\chi,$$

R now being a radius-vector from P to a point on the circumference of the circle.

Similarly

$$\iint\mathrm{LR}^3d\mathrm{R}\,.\cos 2\chi\,d\chi=\int(\tfrac{1}{4}\mathrm{LR}^4+\tfrac{1}{16}\mathrm{R}^4)\cos 2\chi\,d\chi.$$

To calculate the double integrals involved in (33), we shall take as the independent variable the angle $k'Dp$, or ψ; and we have, if c is the radius of the cross-section,

$$R^2 = m^2 + 2cm\cos\psi + c^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{35}$$

$$R^2 d\chi = c(c + m\cos\psi)\,d\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{36}$$

$$R^2\cos 2\chi = 2R^2\cos^2\chi - R^2 = m^2 + 2cm\cos\psi + c^2\cos 2\psi. \tag{37}$$

Hence

$$\iint R^3 dR\,.\cos 2\chi\,d\chi = \tfrac{1}{4}\int R^4\cos 2\chi\,d\chi$$

$$= \frac{c}{4}\int_0^\pi (c + m\cos\psi)(m^2 + 2cm\cos\psi + c^2\cos 2\psi)\,d\psi = \tfrac{1}{2}Am^2, \tag{38}$$

putting A for πc^2. Similarly

$$\iint R^2 dR\,.\cos\chi\,d\chi = \tfrac{1}{2}Am. \quad . \quad . \quad . \quad . \quad . \tag{39}$$

To find the integrals which involve the logarithms, observe that by trigonometrical expansion

$$\log R^2 = \log m^2\left(1 + 2\frac{c}{m}\cos\psi + \frac{c^2}{m^2}\right)$$

$$= \log m^2 + 2\left(\frac{c}{m}\cos\psi - \tfrac{1}{2}\frac{c^2}{m^2}\cos 2\psi + \tfrac{1}{3}\frac{c^3}{m^3}\cos 3\psi - \&c.\right). \tag{40}$$

Hence

$$\iint LR^3 dR\,.\cos 2\chi\,d\chi$$

$$= \tfrac{1}{4}\int LR^4\cos 2\chi\,d\chi + \tfrac{1}{16}\int R^4\cos 2\chi\,d\chi$$

$$= \frac{c}{4}\int_0^\pi L(c + m\cos\psi)(m^2 + 2cm\cos\psi + c^2\cos 2\psi)\,d\psi$$

$$+ \tfrac{1}{16}\int (c + m\cos\psi)(m^2 + 2cm\cos\psi + c^2\cos 2\psi)\,d\psi$$

$$= \frac{A}{8}\left(4m^2 L - 2c^2 + \frac{c^4}{3m^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{41}$$

where L stands now for $\log\frac{8a}{m}$. Finally

$$\iint LR^2 dR\,.\cos\chi\,d\chi = \frac{A}{2}\left(mL - \frac{c^2}{4m}\right). \quad . \quad . \tag{42}$$

(Of course the integrations in ψ are very simple, since $\int_0^\pi \cos n\psi \, . \cos n'\psi \, . \, d\psi = 0$, except when $n = n'$, and then $\int_0^\pi \cos^2 n\psi \, . \, d\psi = \frac{\pi}{2}$. No term in ψ beyond $\cos 3\psi$ contributes to the integral (41).)

Substituting these values in (33), and denoting the resultant conical angle, $\int \Omega d\text{S}$, by Θ, we have

$$\frac{\Theta}{\text{A}} = 2(\pi - \phi) - \frac{\sin\phi}{a}\left(m\text{L} - \frac{c^2}{4m}\right)$$

$$- \frac{\sin 2\phi}{16a^2}\left\{(6\text{L} - 5)m^2 - c^2\left(1 + \frac{c^2}{6m^2}\right)\right\}. \quad (43)$$

This result is susceptible of verification thus. If the components of magnetic force at P per unit pole per unit current are X, Z parallel to OD, OV, we have

$$\frac{d\text{X}}{dx} + \frac{d\text{Z}}{dz} + \frac{d\text{Y}}{dy} = 0,$$

where $\frac{d\text{Y}}{dy}$ is the line rate of increase of force at P perpendicular to the plane, POD, of xz. Y is, of course, zero, but we easily see that

$$\frac{d\text{Y}}{dy} = \frac{\text{X}}{a - m\cos\phi},$$

by taking a position of the point, or pole, close to P along the perpendicular at P to the plane xz. Hence we should have, up to and including terms in $\frac{1}{a^2}$, the result

$$\frac{d\text{X}}{dx} + \frac{d\text{Z}}{dz} + \text{X}\left(\frac{1}{a} + \frac{m}{a^2}\cos\phi\right) = 0. \quad . \quad . \quad (44)$$

The values of X and Z are best found from $\frac{d\Theta}{dm}$ and $\frac{d\Theta}{md\phi}$, which (neglecting A for the moment) are the components of force along and perpendicular to DP.

Now

$$\frac{d\Theta}{dm}=-\frac{\sin\phi}{a}\left(\mathrm{L}-1+\frac{c^2}{4m^2}\right)-\frac{\sin 2\phi}{16a^2}\left\{(12\mathrm{L}-16)m+\frac{c^4}{3m^3}\right\}, \quad (45)$$

$$\frac{d\Theta}{md\phi}=-\frac{2}{m}-\frac{\cos\phi}{a}\left(\mathrm{L}-\frac{c^2}{4m^2}\right)-\frac{\cos 2\phi}{16a^2}\left\{(12\mathrm{L}-10)m-\frac{2c^2}{m}-\frac{c^4}{3m^3}\right\}; \quad (46)$$

also

$$\mathrm{X}=-\frac{d\Theta}{dm}\cos\phi+\frac{d\Theta}{md\phi}\sin\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

$$\mathrm{Z}=\frac{d\Theta}{dm}\sin\phi+\frac{d\Theta}{md\phi}\cos\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

and

$$\frac{d}{dx}=-\cos\phi\frac{d}{dm}+\frac{\sin\phi}{m}\frac{d}{d\phi},$$

$$\frac{d}{dz}=\sin\phi\frac{d}{dm}+\frac{\cos\phi}{m}\frac{d}{d\phi}.$$

The necessary condition (44) is found to be satisfied both for the terms of the first and for those of the second order.

Variable Current-density.—The preceding investigation assumes the current-density to be constant at all points in the cross-section of the wire. If, however, we assume that at any point, Q (fig. 2), it varies inversely as the distance of the point from the axis OV of the current, its value at Q may be written $\sigma\frac{a}{\alpha}$, where σ is a constant and a and α have the meanings already given to them*. If we put, as before, i for the total current traversing the section, we have

$$\sigma a\int\frac{d\mathrm{S}}{\alpha}=i, \quad . \quad . \quad . \quad . \quad . \quad . \quad (48\,a)$$

which gives $\sigma=\frac{i}{\mathrm{A}}$, where A is the area of the cross-section. The resultant conical angle subtended now at P will be

$$\frac{i}{\mathrm{A}}\int\Omega\,\frac{a}{\alpha}d\mathrm{S}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (48\,b)$$

where Ω has the value given in (32); and since, to the second order,

$$\frac{a}{\alpha}=1-\frac{1}{a}(\mathrm{R}\cos\theta-m\cos\phi)+\frac{1}{a^2}(\mathrm{R}\cos\theta-m\cos\phi)^2, \quad (48\,c)$$

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* The necessity for considering this case was pointed out to me by Professor Perry.

to the expression for $\int \Omega dS$ given in (43) must be added the correction

$$\int \Omega \left\{ -\frac{1}{a}(R\cos\theta - m\cos\phi) + \frac{1}{a^2}(R\cos\theta - m\cos\phi)^2 \right\} dS. \quad (48\,d)$$

The term of the second order in Ω in (32) will, of course, contribute nothing to this correction, while the term of the first order in Ω is to be taken with the term

$$-\frac{1}{a}(R\cos\theta - m\cos\phi)$$

only, so that the expression for the correction is

$$2\int (\pi - \theta)\left\{ -\frac{1}{a}(R\cos\theta - m\cos\phi) + \frac{1}{a^2}(R\cos\theta - m\cos\phi)^2 \right\} dS$$
$$+ \frac{1}{a^2} \cdot \int LR\sin\theta (R\cos\theta - m\cos\phi) dS. \quad . \quad (48\,e)$$

Putting $\theta = \phi - \chi$, as before, the only terms of new form that present themselves are

$$2\iint \left\{ -\frac{\sin\phi}{a}\left(1 + \frac{2m}{a}\cos\phi\right). R\chi\sin\chi \right.$$
$$\left. + \frac{R^2}{2a^2}\sin 2\phi . \chi \sin 2\chi \right\} RdRd\chi;$$

and it will be found that

$$\iint R^2 dR . \chi \sin\chi d\chi = \tfrac{1}{3}\int \chi (R'^3 - R^3) \sin\chi d\chi = A\frac{c^2}{8m}, \quad . \quad (48\,f)$$

$$\iint R^3 dR . \chi \sin 2\chi d\chi = \tfrac{1}{4}\int \chi (R'^4 - R^4) \sin 2\chi d\chi$$
$$= A\frac{c^2}{4}\left(1 - \frac{c^2}{6m^2}\right), \quad (48\,g)$$

in which χ has been taken from 0 to ω only, so that in the correction (48 *e*) these must be doubled.

Hence the whole of the correction (48 *e*) amounts to

$$-\frac{\sin\phi}{2a}\frac{c^2}{m} + \frac{1}{16a^2}\left\{ 8c^2(\pi - \phi) - \left(2c^2 + \frac{2c^4}{3m^2}\right)\sin 2\phi \right\},$$

so that (43) is replaced by

$$\frac{\Theta}{A} = 2(\pi - \phi) - \frac{\sin\phi}{a}\left(mL + \frac{c^2}{4m}\right) + \frac{1}{16a^2}\left\{ 8c^2(\pi - \phi) \right.$$
$$\left. - \left[(6L - 5)m^2 + c^2 + \frac{c^4}{2m^2}\right]\sin 2\phi \right\}, \quad (48\,i)$$

the right-hand side being the value of the resultant conical angle subtended by the circuit at the point P. The magnetic potential at P is therefore this right-hand side multiplied by i, the total current traversing the cross-section of the wire.

It will be found that this value of Θ satisfies, both for the terms of the first order and for those of the second, the condition (44).

If the density of the current at any point Q in the cross-section be assumed to vary as any power of the distance of Q from the axis OV, the conical angle subtended at a point near the wire is found just as easily as in the case in which the density is supposed to vary inversely as the distance. Thus, if it is proportional to $\frac{1}{a^n}$, (48 c) will be replaced by

$$1-\frac{n}{a}(\mathrm{R}\cos\theta-m\cos\phi)+\frac{n(n+1)}{2a^2}(\mathrm{R}\cos\theta-m\cos\phi)^2,$$

and we have merely the same terms (48 f), &c., as before.

We see then that when small quantities of the first order, indicated by the fraction $\frac{m}{a}$, are taken into account, the magnetic potential, and therefore the magnetic force, at any point are not the same as if the whole current were condensed into an infinitely thin filament traversing the centre of the wire, D; for, at points, P, close to the wire $\frac{c}{m}$ is nearly unity, and hence the fraction $\frac{c^2}{4m^2}$ is not negligible in comparison with L, unless, indeed, $8a$ is many thousands of times greater than c.

Consider the magnetic force, Z_{E}, just outside the wire at E. This is obtained by putting $\phi=0$ and $m=c$ in $\frac{1}{\mathrm{A}}\,\frac{d\Theta}{md\phi}$; thus, omitting the factor i,

$$\mathrm{Z}_{\mathrm{E}}=-\frac{2}{c}-\frac{1}{a}(\mathrm{L}+\tfrac{1}{4})-\frac{c}{16a^2}(12\mathrm{L}+1),$$

while by putting $\phi=\pi$, $m=c$, we obtain the force, Z_{E}, at B:

$$\mathrm{Z}_{\mathrm{B}}=-\frac{2}{c}+\frac{1}{a}(\mathrm{L}+\tfrac{1}{4})-\frac{c}{16a^2}(12\mathrm{L}+1),$$

whereas the magnetic force at the centre, O, of the circle is of the order $\frac{2\pi}{a}$, and is therefore much less than the force close to the surface of the wire.

Lines of Magnetic Force.—The forms, or approximate equations, of the lines of magnetic force close to the wire may be determined to the second order of small quantities in like manner. Thus, in my previous paper on the Magnetic Field of a Circular Current (*ante*, p. 206) I have shown that if at any point in space in presence of a current running in an infinitely thin circular filament G is the vector potential due to the current, we shall have

$$G\,.\,\alpha=\text{constant}$$

along the line of force, where α is the distance of the point from the axis of the current (*i. e.*, the perpendicular to its plane drawn at its centre). In the case of a point so close to a wire that the current through its cross-section must be broken up into filaments (as in the previous calculation of Θ), the total vector potential at any point is $\int G dS$, and as α is the same for all the filaments, the equation of a line of force is

$$\int G\,.\,\alpha\,.\,dS=\text{constant}.$$

But (Phil. Mag. *ibid.*)

$$G\,.\,\alpha=\{(1+k'^2)K-2E\}\rho,$$

and the current-density in the filament through dS being first supposed constant and equal to $\frac{i}{A}$, the equation of a line of force is

$$\frac{i}{A}\int\{(1+k'^2)K-2E\}\rho dS=\text{constant}. \quad . \quad . \quad (49)$$

With previous notation and approximation, we have

$$K=\log\frac{8a}{R}+\frac{R\cos\theta-2m\cos\phi}{2a}$$
$$+\frac{R^2(L-1-2\cos 2\theta)+8Rm\cos\theta\cos\phi-8m^2\cos^2\phi}{16a^2}, \quad (50)$$

$$E=1+\frac{R^2(2L-1)}{16a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

$$\rho=2a\left(1+\frac{\mathrm{R}\cos\theta-2m\cos\phi}{2a}+\frac{\mathrm{R}^2\sin^2\theta}{8a^2}\right), \quad . \quad . \quad . \quad . \quad (52)$$

$$k'^2=\frac{\mathrm{R}^2}{4a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

where $\mathrm{L}\equiv\log\frac{8a}{\mathrm{R}}$. Hence (49) becomes

$$\frac{4ai}{\mathrm{A}}\iint\left[1+\frac{\mathrm{R}\cos\theta-2m\cos\phi}{2a}+\frac{\mathrm{R}^2\sin^2\theta}{8a^2}\right]\left[\mathrm{L}-2+\frac{\mathrm{R}\cos\theta-2m\cos\phi}{2a}\right.$$
$$\left.+\frac{\mathrm{R}^2(\mathrm{L}+1-2\cos2\theta)+8\mathrm{R}m\cos\theta\cos\phi-8m^2\cos^2\phi}{16a^2}\right]\mathrm{R}d\mathrm{R}\,d\chi, \quad . \quad (54)$$

it being understood that χ ranges from 0 to ω, and therefore ψ from 0 to π, when (as in the calculation of Θ) the independent variable is changed from χ to ψ.

In addition to the integrals (38), (39), (41), (42), the following are now required, and they are easily deduced like the others :

$$\iint \mathrm{L}\mathrm{R}^3 d\mathrm{R}\,d\chi=\frac{\mathrm{A}}{8}\{\mathrm{L}(2m^2+c^2)-2c^2\}, \quad . \quad . \quad (55)$$

$$\iint \mathrm{R}^3 d\mathrm{R}\,d\chi=\frac{\mathrm{A}}{4}(2m^2+c^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

$$\iint \mathrm{L}\mathrm{R}\,d\mathrm{R}\,d\chi=\frac{\mathrm{A}}{2}\mathrm{L}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

$$\iint \mathrm{R}\,d\mathrm{R}\,d\chi=\frac{\mathrm{A}}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

where $\mathrm{L}\equiv\log\frac{8a}{m}$.

With these values (54) gives as the equation of a line of force

$$\frac{\mathrm{L}}{2}-1-\frac{\cos\phi}{4a}\left\{(\mathrm{L}-1)m+\frac{c^2}{4m}\right\}+\frac{1}{16a^2}\left\{\mathrm{L}\left(\frac{m^2}{2}+\frac{c^2}{4}\right)\right.$$
$$\left.+\frac{m^2}{2}-\frac{c^2}{4}-\cos2\phi\left(\mathrm{L}\frac{m^2}{2}-\frac{c^2}{4}-m^2+\frac{c^4}{24m^2}\right)\right\}=\text{const.}, \quad .(59)$$

to the second power of the small quantity or $\frac{c}{a}$.

As a verification, the curve (59) should be at right angles to the curve $\Theta=$const. given by (43), *i. e.*, if we calculate the value of $m\frac{d\phi}{dm}$ for each curve, the one should be the negative reciprocal of the other, as far as small quantities of the second order. This is found to be the case; for, from (59) we have

$$\frac{dm}{md\phi}=\frac{m}{2a}\sin\phi\left(\mathrm{L}-1+\frac{c^2}{4m^2}\right)-\frac{m^2}{8a^2}\sin 2\phi\left\{(\mathrm{L}-2)^2+\frac{c^2}{4m^2}-\frac{7}{48}\frac{c^4}{m^4}\right\},$$

which is $-\frac{md\phi}{dm}$ calculated from the equation $\Theta=$const.

If we denote the ratio $\frac{c^2}{4m^2}$ by λ, the result (59) may be written

$$\tfrac{1}{2}\mathrm{L}-\frac{m}{4a}(\mathrm{L}-1+\lambda)\,.\cos\phi+\frac{m^2}{16a^2}\{\mathrm{L}(\tfrac{1}{2}+\lambda)+\tfrac{1}{2}-\lambda$$
$$-(\tfrac{1}{2}\mathrm{L}-1-\lambda+\tfrac{2}{3}\lambda^2)\cos 2\phi\}=\text{const.} \quad . \quad . \quad (60)$$

Variable Current-density.—Let the current-density now be supposed to vary inversely as the distance from the axis OV. If at any point P in space Γ is the vector potential due to a system of circular currents all having the same axis OV, and if x is the distance of P from this axis, the equation of a line of force is, as has been shown,

$$\Gamma x=\text{constant}.$$

And Γ is the resultant vector potential at P due to a system of currents running in filaments through the wire, the density of the typical current filament through Q (fig. 1) being $\frac{i}{\mathrm{A}}\,.\,\frac{a}{\alpha}$; therefore

$$\Gamma=\frac{4i}{\mathrm{A}}\int\frac{\alpha}{\rho k^2}\{(1+k'^2)\mathrm{K}-2\mathrm{E}\}\frac{a}{\alpha}d\mathrm{S}; \quad . \quad . \quad (61)$$

and since x is the same for all the filaments, in taking Γx we may put x under the sign of integration, so that

$$\Gamma x=\frac{i}{\mathrm{A}}\int\{(1+k'^2)\mathrm{K}-2\mathrm{E}\}\rho\frac{a}{\alpha}d\mathrm{S}. \quad . \quad . \quad . \quad (62)$$

As we require only the terms to be added to (59) by the introduction of the term $\frac{a}{\alpha}$, we may keep only terms of the first order in the expressions for $(1+k'^2)\mathrm{K}-2\mathrm{E}$ and ρ. Thus

$$(1+k'^2)\mathrm{K}-2\mathrm{E}=\mathrm{L}-2+\frac{\mathrm{R}\cos\theta-2m\cos\phi}{2a}, \quad . \quad (63)$$

$$\rho=2a\left\{1+\frac{\mathrm{R}\cos\theta-2\,m\cos\phi}{2a}\right\}. \quad (64)$$

Hence, neglecting the factor $\frac{2ai,}{\mathrm{A}}$ the correction introduced is

$$\iint\left\{\mathrm{L}-2+(\mathrm{L}-1)\,\frac{\mathrm{R}\cos\theta-2m\cos\phi}{2a}\right\}\left\{-\frac{\mathrm{R}\cos\theta-m\cos\phi}{a}\right.$$
$$\left.+\frac{(\mathrm{R}\cos\theta-m\cos\phi)^2}{a^2}\right\}\mathrm{R}\,d\mathrm{R}\,d\chi. \quad . \quad (65)$$

In this the term of the first order is

$$-\frac{1}{a}\iint(\mathrm{L}-2)\,(\mathrm{R}\cos\theta-m\cos\phi)\;\mathrm{R}\,d\mathrm{R}\,d\chi, \quad . \quad . \quad (66)$$

which, by putting $\cos\theta=\cos\phi\cos\chi$, is simply $\mathrm{A}\,\frac{\cos\phi}{a}\cdot\frac{c^2}{8m}$.

The term of the second order is

$$\frac{1}{2a^2}\cdot\iint\{(2\mathrm{L}-4)\,(\mathrm{R}\cos\theta-m\cos\phi)^2$$
$$-(\mathrm{L}-1)\,(\mathrm{R}^2\cos^2\theta-3m\mathrm{R}\cos\theta\cos\phi+2m^2\cos^2\phi)\}\mathrm{R}\,d\mathrm{R}\,d\chi, \quad (67),$$
$$=\frac{1}{2a^2}\cdot\iint\{(\mathrm{L}-3)\mathrm{R}^2\cos^2\theta-(\mathrm{L}-5)m\mathrm{R}\cos\theta\cos\phi$$
$$-2m^2\cos^2\phi\}\mathrm{R}\,d\mathrm{R}\,d\chi, \quad (68)$$

which, by putting $\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\phi\cos 2\chi$, &c., reduces to

$$\frac{\mathrm{A}}{32a^2}\left\{-\mathrm{L}(2m^2-c^2)-7c^2+\cos 2\phi\left(-c^2+\frac{c^4}{3m^2}\right)\right\}. \quad (69)$$

[To prevent error, it may be observed that it is not allowable to put $\cos\theta=\cos\phi\cos\chi$ in any of the equations (63), (64), or (65) until, as in (66) or (67), all multiplications introducing powers of $\cos\theta$ have been performed.]

The terms (66) and (69), after neglecting the factor A, when added to (59) give as the equation of a line of force near the wire,

$$\frac{\mathrm{L}}{2}-\frac{\cos\phi}{4a}\left\{(\mathrm{L}-1)m-\frac{c^2}{4m}\right\}+\frac{1}{16a^2}\left\{\mathrm{L}\left(\frac{3c^2}{4}-\frac{m^2}{2}\right)\right.$$
$$\left.+\frac{m^2}{2}-\frac{15c^2}{4}-\cos 2\phi\left(\mathrm{L}\frac{m^2}{2}+\frac{c^2}{4}-m^2-\frac{c^4}{8m^2}\right)\right\}=\text{const.} \quad (70)$$

As before, we can verify this result by showing that the curve (70) is at right angles to the curve (48 *i*). Taking, as previously, $\frac{c^2}{4m^2}=\lambda$, this equation is

$$\tfrac{1}{2}\mathrm{L}-\frac{m}{4\alpha}(\mathrm{L}-1-\lambda)\cos\phi+\frac{m^2}{16a^2}\{(3\lambda-\tfrac{1}{2})\mathrm{L}+\tfrac{1}{2}-15\lambda$$
$$-(\tfrac{1}{2}\mathrm{L}-1+\lambda-\tfrac{1}{2}\lambda^2)\cos 2\phi\}=\text{const.} \quad (71)$$

It is interesting to observe that the supposition of variable density affects the term of the first order both in the value of Θ and in the constant of the line of force in the same way.

To trace any Line of Force.—With the centre, D, of the cross-section of the wire describe a series of circles, fig. 3, their radii being D*f*, D*g*, D*h*, . . . Then to trace the particular line of force which passes through *f* (suppose) let $Df=m_0$, and let m be the radius of any other of the circles. If ϕ is the angle defining the point in which this latter circle is met by the line of force through *f*, we equate the left-hand side of (60) in case of constant density, and of (71) in case of variable density, to the expression which this left-hand side assumes when m_0 and 0 are put for m and ϕ. Firstly, neglecting the terms of the second order, we have, to determine ϕ, in the case of constant density,

$$\tfrac{1}{2}L-\frac{m}{4a}(L-1+\lambda)\cos\phi=\tfrac{1}{2}L_0-\frac{m_0}{4a}(L_0-1+\lambda_0);$$

and if ϵ is the value of ϕ given by this equation, we can put $2\phi=2\epsilon$ in the terms of the second order. If the term of the second order, $\frac{m^2}{16a^2}\{.\ .\ .\}$, in (60) is denoted by γ, the more correct value of ϕ is obtained from the equation

$$\frac{m}{4a}(L-1+\lambda)(\cos\phi-\cos\epsilon)=\gamma-\gamma_0,$$

where in γ we put 2ϵ for 2ϕ, and in γ_0, of course, $2\phi=0$.

As a numerical example, let the wire in fig. 1 form a circle 20 millim. in diameter, *i. e.* OD in fig. 2 is 10 millim.; and let the diameter of its cross-section be 2 millim. *i. e.* DB= 1 millim. in fig. 2.

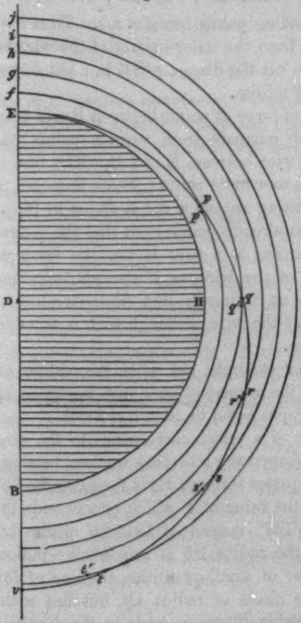
In fig. 3 let EHB represent half the cross-section of the wire, its centre being D, and let the centre, O (see fig. 1), be in the production of the line DE at a distance 10 millim. above D, while DE=1. Let a series of circles be described round D with radii D*f*, D*g*, D*h*, . . . equal to 1·1, 1·2, 1·3, 1·4, 1·5 millim., and suppose that we trace the line of torce which touches the wire at E. If we calculate the angle ϕ

which defines the point in which this line cuts the circle of radius Df, we find, taking only terms of the first order, $\epsilon = \mathrm{ED}p'$, or

$$\epsilon = 63^\circ\ 21'.$$

This gives the point p'; but taking the terms of the second order the angle becomes 62° 37′, which gives the true point,

Fig. 3.



p, by means of the angle EDp. Similarly the point, q', in which, if only terms of the first order were taken account of, the line of force would cut the next circle (of radius Dg) is defined by the value $\epsilon = 90° \, 14'$, which, being corrected by the terms of the second order, becomes 89° 9′, and the corresponding point is q.

In like manner the points r', s', t' given by terms as far as the first order correspond to values of ϵ equal to

$$112° \, 48' \, ; \; 134° \, 26' \, ; \; 165° \, 41',$$

which are corrected into

$$111° \, 40' \, ; \; 133° \, 13' \, ; \; 163° \, 17',$$

the corresponding points being r, s, t. Thus the line of force which starts from the inner surface of the wire is E$pqrst$, and it is found to cut the diameter EB in a point v such that Dv is about 1·52 millim.

The lines of force at points between B and v are incomplete curves which emanate from various points on the surface EHB of the wire between E and B. The line of force at B itself reduces to a mere point. These lines can, of course, be traced by putting m_0 and π for m and ϕ in (60).

It has been already pointed out that the magnetic effect of a current running in a wire is not the same, at points near the surface of the wire, as if the whole current were concentrated in an infinitely thin filament running along the central line of the wire, although such is sometimes assumed to be the case.

Let us, for example, see what the values of ϕ, or of ϵ, would be in the numerical case just discussed if we assume that c can be put equal to zero, *i. e.* $\lambda = 0$.

The value of ϵ which corresponds to the circle of radius 1·1 millim. (supposing still that we are tracing the line of force which passes through E) is found to be 66° 15′ instead of 63° 20′; the value of ϵ which corresponds to the circle of radius 1·4 is 142° instead of 134° 26′, while that which corresponds to the radius 1·5 is impossible—indicating that, to the first order of small quantities, the line of force does not intersect the circle of radius Dj, but lies within it. Thus there is a notable difference made in the results by assuming that the whole current can be concentrated in a line running through D.

If $a = \infty$, *i. e.*, if the current runs in a straight wire, the conical angle and the constants of the lines of force are the same as if $c=0$, and therefore *such concentration of a current along the central line of the wire is allowable only when the wire is straight, or when the curve into which it is bent has a very large radius of curvature.*

In the case of variable current-density the equation determining ϵ is

$$\tfrac{1}{2}\mathrm{L}-\frac{m}{4a}(\mathrm{L}-1-\lambda)\cos\epsilon=\tfrac{1}{2}\mathrm{L}_0-\frac{m_0}{4a}(\mathrm{L}_0-1-\lambda_0);$$

and applying this to the same numerical case, we find that p', q', r', s' are determined by the angles

68° 49′; 98° 10′; 123° 4′; 152° 4′,

while the position of t' becomes imaginary. This shows that the supposition of variable density brings the lines of force closer to the surface, B, of the wire—as we should expect *à priori*. It is not considered necessary to draw a separate figure for the case of variable density, since the forms of the lines of force and the method of drawing are sufficiently illustrated by the case of assumed constant density.

Initial or rapidly alternating Currents.—The magnetic potential and the forms of the lines of force will not be the same when the current has become steady as they were in the initial stage of the current, because, just at starting, the current is confined to the surface of the wire. If we can assume that when the current is entirely superficial its density (or its infinitesimal depth below the surface of the wire) is constant, the magnetic potential at any point P and the constant of the line of force can be obtained by subtracting from the value of the potential in (43) its value when c is replaced by $c-\Delta c$, and a similar subtraction from (59). Thus, the magnetic potential is the right-hand side of (43) multiplied by i, the total current in the cross-section. If δ is the density of this current, $i=\pi c^2\delta$, and if q is the total superficial current $(=2\pi c\delta\,.\,\Delta c)$, the magnetic potential becomes

$$q\Big[2(\pi-\phi)-\frac{\sin\phi}{a}\Big(m\mathrm{L}-\frac{c^2}{2m}\Big)$$
$$-\frac{\sin 2\phi}{16a^2}\Big\{(6\mathrm{L}-5)m^2-c^2 2+\Big(\frac{c^2}{2m^2}\Big)\Big\}\Big].$$

From (49) it appears that the constant of the line of force due to the current in the whole cross-section is (59) multiplied by i, or by $\pi c^2 \delta$. When this multiplication is effected, the differentiation with respect to c is to be performed, as in the case of the potential.

Of course the preceding discussion applies to the case of a circular vortex ring in a perfect fluid, the velocity of a particle at any point of the fluid being the analogue of the magnetic force. See Basset's 'Hydrodynamics,' vol. ii. chap. xiv. Mr. Basset assumes that for a vortex ring the magnitude of the cross-section is negligible, so that the concentration of the ring along its core filament is allowable; but, as we have seen, this requires that the ring must be an extremely thin one, and in most cases of the coiling of wires for the conveyance of currents the assumption would be inadmissible; for, the ideal vortex filament required by the assumption (infinitely thin as compared with the radius of its aperture) finds but a very coarse representative in any coil of wire. We have already given the values of the magnetic force at the centre, O, of the curve formed by the wire and at the points, E, B, just outside its surface; these are the analogues of the velocity of the (irrotationally moving) fluid at these points in the case of a vortex ring. The velocity with which the ring itself moves forward is given by Mr. Basset as equal to $\frac{f}{2\pi a}\left(\log \frac{8a}{c} - 1\right)$, where f is the strength of the vortex (product of the cross-section and molecular rotation), while the velocity of the fluid at O is $\frac{f}{a}$; so that the ratio of the forward velocity of the ring itself to the velocity of the fluid at the centre, O, of its aperture is

$$\frac{\log \frac{8a}{c} - 1}{2\pi},$$

which, as Mr. Basset says, is "large" in the case supposed (*i. e.*, a very thin ring). We must observe, however, when comparing actual electrical coils with fine vortex rings, that for a vortex ring for which a is 1000 times c (which would ordinarily be considered as a "fine" ring), this ratio is not very large: it amounts only to 1·27; while for a vortex ring the radius of whose aperture is 100 times that of its cross-section, this ratio is only ·9, *i.e.*, the ring moves more slowly than the fluid at its centre.

XXV. *A New Form of Influence-Machine.*

By JAMES WIMSHURST[*].

IN April 1891 I had the honour to submit to this Meeting a very useful form of experimental Influence-machine, by means of which I was able to show that almost every combination of glass and metal, and also that plain glass disks, when moved and suitably touched, were capable of producing a flow of electricity.

It is one of those combinations, somewhat modified and extended, which I have now the further pleasure of bringing to your notice.

The machine consists of two disks of plate-glass, each of 3 ft. 5 in. diameter and ¼ inch thickness, mounted about ¾ inch apart on one boss and spindle. This spindle is driven by means of a handle, and the disks rotate in one direction.

In the space between the disks are fitted four vertical slips of glass, two being situated on the right-hand side of the machine and two on the left-hand.

The vertical edges of the slips which come nearest the spindle are cut to an angle, leaving a rather wider opening at the circumference than at the centre of the disk. Upon each slip is a brush, A, and also an inductor, the brush and the inductor being metallically connected; the brushes are made of fine wire, and touch very lightly on the inner surfaces of the disks. The glass slips slide into place by means of suitable grooves at their top and bottom ends; they may be removed and replaced readily at pleasure.

The essential parts are fitted together within a glass case, and in all respects the machine is so constructed as to be useful for experimental work. The limited size of the case causes the insulating distances between part and part to be small, hence the length of the sparks is reduced. All the surfaces are coated with shellac, and when the brushes are new and in proper condition the machine is self-exciting.

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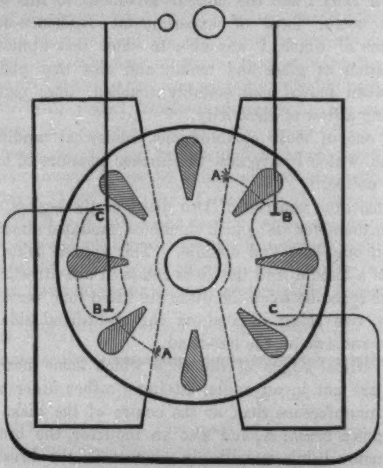
[*] Read June 23, 1893.

The charges are not subject to reversal when the terminals are opened beyond the striking distance, for then the whole of the induced charges pass by way of the neutralizing brushes C, C. Another feature is that the neutralizing current may also be broken without reducing the excitement, but then the charges alternate from positive to negative with each half revolution of the disks.



A, A. Brushes connected metallically to the inductors.
B, B. Collecting-brushes connected with terminals.
C, C. Brushes connected with the neutralizing circuit.

An approximate measure of the efficiency of the machine is seen by the following experiments.

The glass throughout was held free from electrical excitement, and the disks were rotated sixty times per minute ; the actuating cause was then removed, and the disks came to rest after making forty-seven revolutions. This measures the friction of the machine.

The disks were then similarly turned, but the electrical charges were allowed to collect; it was then found that the disks came to rest after twenty-three revolutions. The friction of the machine, therefore, is about equal to the repulsion of the electrical charges.

Leyden jars having eight square inches of tinfoil in each coating were then connected to the terminals, and the terminals set 3½ in. apart: the disks were turned to the same number of revolutions and then left to come to rest, which they did after producing eighty sparks.

The disks were then turned and the number of sparks counted in relation to each revolution. This gave seven sparks of 1 in. length, five of 2 in., four of 3 in., and three of 4½ in. length.

Eight metal sectors were then added (total 112 square inches on each disk): with them the self-excitement becomes so free that, before any circuit was made, presenting the fingers to the disks caused electrical discharges.

When the circuits are made between the several brushes the disks appear to be seething with electricity, and the charges shoot out for a considerable part of their circumference, but chiefly at the top and the bottom, where they are not covered by the vertical glass slips, for the change of the charges from + to − takes place upon the disks just as they pass the edge of the slip. Improvements in this respect might be made, but the present arrangement has the advantage of giving permanence to the respective charges.

It will be noticed that strips of tinfoil are placed over the receding surfaces of the glass slips. They were intended to afford the means for collecting and carrying to earth any part of the charges from the inductors, but the earth connexion has not been necessary.

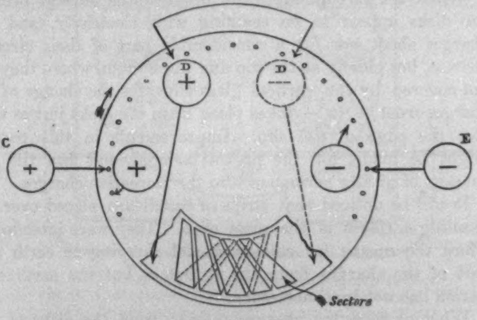
When at work the charges passing from the disks to the brushes may be heard some considerable distance; they resemble the beating of the floats of a paddle-wheel.

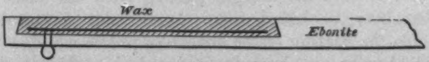
The charges are, however, reduced about 25 per cent. by the addition of the sectors.

Further tests have been made by fixing a pulley, 3 in. in diameter, to the overhanging end of the spindle, and suspending therefrom by a cord a weight of 15 lbs. The fall of this weight through three feet produced seventeen revolutions when the disks were not excited and ten revolutions when excited. The same Leyden jars were then connected, and it was found that the fall of the weight through three feet produced twenty-eight sparks of 3½ in. length. Reducing this to the terms of work in relation to sparking length, and omitting friction of the machine, it will be seen that 1 lb. weight falling through rather less than twelve inches produces a spark of 3½ in. length. It must not, however, be forgotten that a considerable amount of electricity is also passing by way of the neutralizing circuit.

XXVI. *An Influence-Machine.* *By* W. R. Pidgeon[*].

If we follow the action of any single sector on a disk of one of Mr. Wimshurst's beautiful machines, we find it goes through the following electrical changes :—Suppose it





Section of Disk.

to be just leaving the positive collector C, it comes into a strong positive field produced by the other disk, and while in this field is earthed by the first neutralizing brush D and becomes charged negatively. Passing on it induces a positive charge on the other disk at the next neutralizing brush D′, and ultimately meets a similar negative charge, borne forward by the other disk, at E, where it discharges into the negative collector. I do not mean that this is a full explanation of all that takes place, but I hope by its means to show the lines along which I worked in designing the machine before you. I thought that what had to be done was:—1st. To make the capacity of each sector on the disk as great as possible while being charged at D, and as small as possible while being discharged at E, so that the amount of electricity displaced by each sector should be as large as possible. 2nd. To prevent any leak-back of electricity from sector to sector as they are entering, or leaving, the different fields of induction. 3rd. To make the sectors large and numerous so as to increase the capacity of the machine. And it was with these objects in view that I built the machine which I will now describe. The disks are made of ebonite $\frac{5}{8}$ inch thick and are recessed as shown in the drawing, $\frac{3}{8}$ inch deep × $5\frac{3}{8}$ inches wide. There are 32 sectors on each disk made of thin sheet-brass, and having short brass wires soldered to them on to which the small brass collecting-knobs are screwed. The sectors were arranged in the recess as shown on the drawing, the wires attached to them were passed through holes in the disk, and the knobs screwed on from the back. The recess was then filled up to slightly above the level of the face of the disk with melted wax composed of equal parts by weight of paraffin and rosin, and when cold the excess wax was turned off in the lathe. Each sector is thus entirely imbedded in an insulating material, the only exposed part, from which the charge might brush back from sector to sector, being the small brass knob. In order to minimize this possible leak the sectors are set at an angle to any radius, as shown on the drawing, so that a sector on one disk does not pass a sector on the opposite one suddenly, but does so with a sort of shearing motion requiring four times the angular movement of the disks for the sectors to clear one another than would be needed were they placed radially. By the time, moreover, that any one sector has entirely entered a field of induction, the next following sector is three parts into the field, and the next behind that again is half in, while the third is one fourth the way in, and the fourth just entering, so that the maximum difference of potential between any two neighbouring knobs is reduced to one fourth of what it would be were the sectors arranged radially. This angular arrangement of the sectors naturally necessitates the collecting and neutralizing brushes being displaced from the positions they would occupy were the sectors radial so as to touch the latter at the right moment. In order to increase the capacity of each sector while receiving its charge, two stationary inductors are placed opposite each disk, one at each of the points where the sectors on that disk are earthed. Each inductor consists of a sheet of tinfoil imbedded in wax and supported on a disk of ebonite, and is charged from a pin-point connected to it through an ebonite tube; this pin-point stands opposite to the outer knobs on another part of the large disk, where their sign and potential are such as are required to charge the inductor. Each sector at the moment of being earthed is thus placed between two similarly charged inductors,—the sectors on the opposite disk on the one side and the fixed inductor on the other, each of which is large in comparison with itself; its capacity is thereby greatly increased and it is enabled to carry forward a much larger charge of electricity than it can when the stationary inductors are removed.

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[*] Read June 23, 1893.

The effect produced on the output of the machine by using these stationary inductors is remarkable, for though they practically make no difference to the length of the sparks, they greatly increase their apparent thickness; and with other things precisely similar they increase their frequency three-fold. To compare the output of the machine with and without the stationary inductors, I counted the number of sparks which overflowed a Leyden jar, and the revolutions of the disks, for 30 seconds, both when using the inductors and when they were taken off, and I was careful, in every case, to keep the speed of the disks as nearly as possible constant. I thus found that a 25-ounce Leyden jar overflowed 19 times in 30 seconds without inductors and 54 times with them; or, making a slight correction for the revolutions of the disks being less in the second case, the output was increased as

3 to 1 by the use of the inductors. The explanation of this greatly increased output is, I think, to be sought in the fact that connecting the positive and negative terminals of this machine does not stop its action, as it does in the case of the Wimshurst machine, and that it is therefore ready to start sparking, with almost full vigour, the moment they are separated. For when the inductors are removed, and two or three Leyden jars connected to the terminals, it is perfectly easy to note the discharge of the disks with each spark, and the time they take to recover, slowly at first and very rapidly at last, depending on the size of the condensers they are charging; but when the fixed inductors are replaced the recovery is very much more rapid, especially in its early stages, owing I think to the fact that the inductors apparently do not lose their charge with each spark, and therefore, at once, start up the action of the machine.

With a view to decrease the capacity of each sector at the moment of its being discharged into the collector, I put on another set of inductors, one opposite to each of the main collecting brushes, and so arranged matters that they could be charged either from the neighbouring inductor or from the collector itself. In this way each sector at the moment of discharging itself into the collector is almost surrounded by a field of the same sign as itself, and I hoped by this means to squeeze more of its charge out of it as it passed. I was at first, however, disappointed to find that this second set of inductors made little if any difference to the output of the machine, whether they were connected to the other inductors or to the main collectors. But on arranging them to collect their own charge directly from the small knobs on the revolving disks, at a point just before the main collecting-brushes, they became of value and increased the efficiency of the machine nearly $\frac{1}{5}$th.

By the kindness of Mr. Wimshurst and Dr. Lewis Jones I have been enabled to compare the output of my machine with that of a Wimshurst having eight 15-inch plates and also with one having two 16-inch plates. I find that, area for area, and without using the stationary inductors, the efficiency of my machine is to that of the Wimshurst as 3 to 2; and with the inductors in use it is as 5·64 to 1. That is, from 5 to 5$\frac{3}{4}$

times the area of plate is required to pass the collector in the Wimshurst machine to that which must pass it in this machine to produce the same result ; or, to give the figures as we took them together :—

	sq. ft.
Mr. Wimshurst's 8 plate 15-in. diam. machine requires of area to pass the collector per spark.	97·07
Mr. Jones's 2 plate 16-in. diam. Wimshurst requires	87·83
W. R. Pidgeon's 4 plate 24-in. diam. machine, without inductors	63·64
” ” ” ” with one set of inductors on	21·117
” ” ” ” with both sets of inductors on	17·2

I feel I ought to mention a mistake I fell into, while working on this machine, which nearly made me think that fixed inductors were of no use. When first trying them I used brushes, instead of a single point, to collect their charge, and was disappointed to find the output of the machine reduced to almost *nil*, because, as I ultimately discovered, the brushes formed a bridge between the sector knobs and allowed the charge to flow back from sector to sector, instead of being carried forward to the collector. Another difficulty I have not been able to get over, and one which greatly decreases the actual output of the machine, is caused by the warping of the ebonite disks, which, although $\frac{5}{8}$ inch thick, will not stand true after being turned up, but move just sufficiently to throw them out of balance and prevent their being driven at anything approaching a high speed. The ebonite, moreover, is not a sufficiently good insulator, and not only prevents my getting a long spark but also reduces the possible efficiency of the machine by becoming an increasingly good conductor as the potential rises.

DISCUSSION.

In a written communication Prof. O. Lodge said his assistant, Mr. E. E. Robinson, constructed a machine on lines similar to Mr. Pidgeon's, a few months ago, and had now a large one nearly completed. Mr. Robinson's fixed inductors are carried on a third plate fixed between the two movable ones. The sectors are quite small, and neither they nor the inductors are embedded. On close circuit the machine gives a large current ($\frac{1}{10000}$ ampere) and on open circuit exceedingly high potentials. In Dr. Lodge's opinion Mr. Pidgeon attaches too much importance to his sectors and their shape.

Mr. J. Gray wrote to say that stationary inductors enclosed in insulating material would probably give trouble at high voltages, because of the surface of the insulator becoming charged with electricity of opposite sign to that on the inductor. He suggested that this might explain why Mr. Pidgeon could not obtain very long sparks.

Prof. C. V. Boys enquired as to how far the wax made insulating union with the ebonite, for, if good, glass might possibly be used instead of ebonite. He greatly appreciated the design of Mr. Pidgeon's machine.

After some remarks by the President, on the great advances which had been made, Mr. Pidgeon replied, and Mr. Wimshurst tried some further experiments with a small experimental machine.

PROCEEDINGS

OF

THE PHYSICAL SOCIETY

OF LONDON.

JANUARY 1894.

XXVII. *Long-Distance Telephony.* *By* Prof. PERRY, *F.R.S., assisted by* H. A. BEESTON[*].

WHEN resistance, capacity, self-induction, and leakage are taken into account, this subject is one of considerable difficulty. It is given to very few men to be able to discuss complicated mathematical formulæ without making mistakes—the proceedings of Scientific Societies possess many such mistakes detected and undetected—and consequently I instruct my students to experiment with their formulæ, using numerical values for their variables. The consideration of the most general problem of long-distance telephony, involving certain terminal conditions, has been taken up by Mr. Heaviside; but the ordinary mathematical physicist must find great difficulty in understanding the investigation (Phil. Mag. January 1887). Some of my students have recently obtained numerical results, neglecting the terminal conditions, which seem to me to be very instructive, and I think that the Tables will have a permanent value.

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[*] Read June 23, 1893.

As a matter of fact, the line is supposed to be of infinite length, and we consider the state of a signal as it gets farther and farther away from the origin.

By comparing the current c at a section x centimetres from the origin with the current at $x+dx$, and properly disposing of the difference, we arrive at the equation :—

$$\frac{d^2c}{dx^2}=kl\frac{d^2c}{dt^2}+(kr+sl)\frac{dc}{dt}+src, \quad . \quad . \quad . \quad . \quad (1)$$

where (per unit length of conductor) k is the capacity, r the resistance, l the self-induction, and s the leakage conductivity. The solution which suits telephonic conditions is

$$c=a\epsilon^{-hx}\sin(pt-gx),$$

where

$$\sqrt{\frac{kpr}{2}}\sqrt{\sqrt{\left(1+\frac{p^2l^2}{r^2}\right)\left(1+\frac{s^2}{k^2p^2}\right)}\mp\left(\frac{pl}{r}-\frac{s}{pk}\right)}$$

gives the value of h if the minus sign is taken, and gives the value of g if the plus sign is taken, and $c=a\sin pt$ is the current at the origin. Of course $p=2\pi f$, where f is the frequency. Any number of such functions of any frequencies may exist simultaneously.

Two conditions must be satisfied in telephony. Taking the shrillest and gravest notes of the human voice as being of frequencies f and f', and taking therefore currents of these frequencies :—let X be the distance at which the ratio of the amplitudes of the shrill and grave currents is increased by $1/m$th of itself ; let Y be the distance at which one of the currents has altered in lag behind the other by $1/n$th of the periodic time of the more rapid one ; then it is easy to see that

$$X=1\Big/\{m(h-h')\}\,*,$$

$$Y=2\pi\Big/\left\{np\left(\frac{g}{p}-\frac{g'}{p'}\right)\right\}.$$

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* This is approximate. If m is not large, the true expression ought to be used,

$$X=\log_e\left(1+\frac{1}{m}\right)\Big/(h'-h).$$

The letters with dashes indicate that p' or $2\pi f'$ must be taken instead of p. It is easy to see that X and Y become infinite if $\frac{l}{r}=\frac{k}{s}$.

I do not know what values of m and n would produce confusion of sound in the telephone. But as an exercise we have taken $m=4$ and $n=6$. We have also taken $p=6000$, $p'=600$.

For the first French Atlantic Cable the capacity and resistance were 0·43 microfarad and 2·93 ohms per nautical mile, so that

$$k=2{\cdot}3215\times10^{-12} \text{ farads per centim.}$$

$$r=1{\cdot}582\times10^{-5} \text{ ohms per centim.}$$

Mr. Beeston has calculated the distances X and Y, the lesser of which may be taken as the limiting distance for good telephony for various values of l and s.

TABLE I.—Limiting distances X in millions of centimetres for various amounts of leakage and self-induction. (One million centimetres are equivalent to about six miles.)

Values of $s\times10^{10}$.	Values of $l\times10^{10}$.						
	0	2·6373	26·373	79·118	131·863	184·61	263·73
0	0·983	1·054	1·963	5·169	10·178	17·500	33·839
·01	...	...	1·961				
·10	...	...	1·969				
1	...	...	1·999				
5	1·049	1·130	2·187	6·339	13·944	41 532	75·59
10	1·130	1·224	2·356	8·462	21·98	59·03	328·2
20	1·319	1·444	3·357	18·009	128·17	very large	131·65
40	1·754	1·965	6·250	390·11	67·95	18·94	8·75
70	2·527	2·935	17·671	39·73	7·727	4·161	2·573
100	3·447	4·134	72·03	10·00	0·650	2·111	1·617
150	5·314						
200	7·536						
250	10·090						

Table II.—Limiting distances Y in millions of centimetres for various amounts of leakage and self-induction.

Values of $s\times10^{10}$.	Values of $l\times10^{10}$.						
	0	2·6373	26·373	79·118	131·863	184·61	263·73
0	1·459	1·484	1·778	2·665	3·719	4·996	7·239
·01	...	...	1·783				
·10	...	...	1·800				
1	...	...	1·893				
5	1·889	1·927	2·424	4·142	6·674	10·472	20·29
10	2·426	2·482	3·297	6·639	14·130	28·88	116·36
20	3·752	3·874	5·796	18·700	82·59	very large	75·12
40	7·292	7·649	15·152	566·36	76·55	18·184	7·214
70	15·216	17·342	57·965	76·304	11·493	5·266	2·815
100	27·078	30·147	283·49	22·70	0·354	3·267	1·810
150	58·35						
200	106·43						
250	170·36						

It will be noticed that although X and Y are derived in different ways, by taking certain values for m and n they could be made much the same in value, and altering s or l seems to produce the same sort of effects on X and Y.

Mr. Beeston has drawn curves from the calculated numbers, but these need not be published.

If there is no self-induction, increasing the leakage increases the distance to which we can telephone. If there is no leakage, increasing the self-induction increases the distance.

When the amounts of s and l are not too great, increasing either increases the distance. These and other important facts are visible in the tables.

Without such tables as these and this method of study it would be almost impossible for the average mathematician to make anything of his mathematical results. Thus, for example, when equation (1) applies to such a function as $\sin pt$ it is just the same as

$$\frac{d^2c}{dx^2}=\left(kl-\frac{sr}{p^2}\right)\frac{d^2c}{dt^2}+(kr+sl)\frac{dc}{dt}\,. \quad . \quad . \quad . \quad (2)$$

Hence we see that the effect of leakage is to *diminish* the self-induction by the amount sr/kp^2, and to increase the resistance by the amount sl/k. But it is easy to see that if we diminish self-induction or increase resistance we do harm in telephony, and yet this kind of diminution through leakage does good. On going into the matter carefully, it is seen that it is the p being in the denominator of sr/kp^2 which produces the good effect. In fact, if l and s are small, taking $p=6000$, $p'=600$, we find

$$X \propto 1 \Big/ \sqrt{kr}\left\{1-4246\frac{l}{r}-\frac{s}{3800k}\right\}.$$

So that increasing s or l produces a good effect. Having found the mathematical reason we have not far to go to find the physical reason.

It is evident from the tables that if we had no leakage we could completely get rid of the evil effects of capacity by introducing self-induction. It is also evident that if we had no self-induction, we could completely get rid of the evil effects of capacity by introducing leakage. But when there is some leakage and some self-induction, we can in practice only mitigate the evil effect of capacity; for it is obvious that, although certain values of l and s give infinite distances, doubling or halving these values produces enormous diminution in distance, and such a constant of a cable as s may alter very greatly.

About fifteen years ago, with Prof. Ayrton I made many experiments on signalling through bare copper wires lying at the bottom of the water in the moat of Yedo in Japan. Here k and s were both very great. We had much less success than we expected, and we abandoned, perhaps too readily, our idea of a very cheap submarine cable.

The following tables are of general application. Let the numbers given in Table III. be divided by the value of $\sqrt{kr}$ for any cable or conductor of a telephonic line, and let them also be divided by the value of m which is considered suitable[*], and they will become the limiting distances X in centimetres for that conductor, for the various values of $\frac{l}{r}$ and $\frac{s}{k}$ given.

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[*] It is more correct to say that the numbers are to be multiplied by $\log_e\left(1+\frac{1}{m}\right)$.

TABLE III.

Values of s/k.	Values of $10^6 \times l/r$.						
	0	1·667	16·67	50	83·33	116·7	166·7
0	0·0267	0·0286	0·0533	0·1404	0·2765	0·4753	0·9191
215·4	0·0285	0·0307	0·0594	0·1722	0·3787	1·1281	2·053
600	......	......	......	......	......	......	∞
430·8	0·0307	0·0332	0·0640	0·2298	0·5970	1·6034	8·913
857	......	......	......	......	......	∞	
861·5	0·0358	0·0392	0·0912	0·4891	3·4732	2755·0	3·576
1200	......	......	......	......	∞		
1723	0·0476	0·0534	0·1698	10·596	1·8455	0·5145	0·2377
2000	......	......	......	∞			
3015	0·0686	0·0797	0·4799	1·0792	0·2099	0·1130	0·0699
4308	0·0936	0·1123	1·9564	0·2717	0·0176	0·0573	0·0439
6000	......	......	∞				
60000	......	∞					

TABLE IV.

Values of s/k.	Values of $10^5 \times l/r$.						
	0	1·667	16·67	50	83·33	116·7	166·7
0	0·0531	0·0540	0·0646	0·0969	0·1352	0·1813	0·2633
215·4	0·0687	0·0700	0·0881	0·1506	0·2427	0·3808	0·7379
600	......	......	......	......	......	......	∞
430·8	0·0882	0·0903	0·1199	0·2414	0·5139	1·0502	4·231
857	......	......	......	......	......	∞	
861·5	0·1364	0·1409	0·2108	0·6800	3·0033	111056	2·732
1200	......	......	......	......	∞		
1723	0·2652	0·2781	0·4377	20·596	2·7837	0·6613	0·2624
2000	......	......	......	∞			
3015	0·5533	0·6306	2·1079	2·775	0·4179	0·1915	0·1024
4308	0·9847	1·0963	0·1031	0·826	0·0129	0·1188	0·0658
6000	......	......	∞				
60000	......	∞					

Let the numbers in Table IV. be divided by the value of $\sqrt{kr}$ for any conductor of a telephonic line, and let them also be divided by the value of n which is considered suitable, and they will become the limiting distances Y in centimetres for that conductor, for the various values of $\frac{l}{r}$ and $\frac{s}{k}$ given.

Another way of putting the results in these tables is this :—Let K be the whole capacity of the line in farads, R its resistance in ohms, L its whole self-induction in secohms, S the whole leakage conductivity in mhos : L/R and S/K are the same as the l/r and s/k of the tables. For given values of these find the number in either table ; square it and divide by any chosen m^2 or n^2 ; this gives the product of the whole capacity K and the whole resistance R of the conductor.

Of course in all lines which have the same values of L/R and S/K, the product KR is constant. I must again draw attention to the fact that we have neglected the terminal conditions.

I have not hitherto said anything about the amplitude of the current ; in fact the receiving apparatus has been supposed to be infinitely delicate. It is obvious that $1/h$ is the

TABLE V.

Values of s/k.	Values of $\frac{e}{r} \times 10^5$.						
	0	1·667	16·67	50	83·3	116·7	166·7
0	·0127	·0167	·0197	·0314	·0402	·0476	·0567
215·4	·0124	·0130	·0188	·0283	·0340	·0389	·0331
430·8	·0122	·0128	·0177	·0257	0295	·0315	·0329
861·5	·0118	·0123	·0167	·0218	·0233	·0236	·0232
1723	·0110	·0114	·0145	·0166	·0164	·0158	·0147
3015	·0099	·0102	·0121	·0124	·0115	·0106	·0096
4308	·0091	·0093	·0105	·0100	·0090	·0081	·0072
6462	·0079						
8616	·0071						
10770	·0065						

distance in which the amplitude becomes $1/\epsilon^{\text{th}}$ of its initial amount. If the numbers in Table V. be divided by $\sqrt{kr}$ for any line, they give the distances in which a current of frequency 955 per second has its amplitude *halved*.

The effect of leakage is in every case to diminish the amplitude of the current, making it necessary to have more sensitive receiving-instruments or more powerful sending-instruments.

If some of our clever mathematicians would for a while put aside the ambition to write original papers and would give us in one paper, however long, an exposition of Mr. Heaviside's views on the subject, he would confer great benefits upon the average electrician. Mr. Heaviside can discover new truths, and we all believe in his results when we understand them, but he seems unable to lower his reasoning to our mathematical levels. Since writing this paper I have tried to understand Mr. Heaviside's numerous papers on this subject, but I am sorry to say that I am not yet able to express a certain opinion as to the practical value, or want of value, of the preceding tables.

DISCUSSION.

Mr. Blakesley said that some eight years ago he discussed the subject when capacity and resistance were alone considered, and now pointed out that when self-induction and leakage were introduced, the equations were still of the same form. He also suggested how terminal conditions on lines of finite length might be easily taken into consideration.

Prof. Perry, in reply, said the introduction of self-induction and leakage rendered the calculations much more laborious, and that the terminal conditions were much more complicated than Mr. Blakesley supposed.

XXVIII. *Two Lecture-Room Experiments, by* W. B. CROFT, *M.A.** (Abstract.)

MR. W. B. CROFT, M.A., showed two Lecture-room Experiments. One, on "The Rings and Brushes in Crystals," was performed by a very simple apparatus in two ways. In the first a bundle of glass plates was used as polarizer, and a Nicol prism as analyzer. When a Nicol could not be conveniently obtained, a glass plate could be used as a reflecting analyzer. For a convergent system two glass card-counters were used, the crystal being placed between them. Very good results were produced by this simple apparatus.

In the second arrangement the crystal was placed on the eye-piece of a microscope (whose objective was removed), and covered by a tourmaline. On reflecting light up the tube by means of a piece of glass held at the proper angle, excellent results were obtained.

Another experiment, on "Electric Radiation in Copper Filings," was similar to those described by Dr. Dawson Turner at the Edinburgh meeting of the British Association. A battery, galvanometer, and glass tube containing copper filings were joined in series. Under ordinary circumstances no current passed, but immediately an electric spark was produced by an electric machine many feet away, the galvanometer was violently deflected, and remained so until the tube was tapped. On trying different materials, aluminium and copper seemed about equal, but iron not so good. Carbon allowed the current to pass always.

DISCUSSION.

Prof. Minchin said the phenomena were strikingly like those exhibited by his "impulsion cells," for the moment a spark passed, even at a distance of 130 feet, they became sensitive to light. Very minute sparks were capable of producing the change, but by adding capacity to the sparking circuit the effect could be greatly modified. Replying to a question from Mr. Rimington, he said the change was due to electro-magnetic vibrations, and not to light emitted by the sparks.

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* Exhibited October 27, 1893.

Mr. Blakesley enquired if lengthening the sparks produced greater effect on the copper filings.

Mr. Lucas asked if the resistance of a tube ever became infinite again if left for a long time.

In reply, Mr. Croft said the current sometimes passed before the spark actually occurred between the knobs. He had not left tubes for very long, and had not found the resistance reappear without tapping.

XXIX. *On the Separation of Three Liquids by Fractional Distillation.* *By* Professor F. R. Barrell, *M.A., B.Sc.,* G. L. Thomas, *B.Sc., and* Professor Sydney Young, *D.Sc., F.R.S., University College, Bristol*[*].

It is well known that the separation by fractional distillation of two substances which are miscible in all proportions and which do not—like propyl alcohol and water or formic acid and water—form mixtures of constant boiling-point, is usually a simple matter if there is a considerable difference in their boiling-points. The facility with which the separation can be effected depends, in fact, chiefly on this difference.

When, however, we have to deal with a mixture of three substances, the difficulty is greatly enhanced, and if the boiling-points are not far apart it may be almost, if not quite impossible to separate any quantity of the middle substance in a state of purity by the ordinary methods of fractional distillation.

The variation in the composition of the distillate from a mixture of two substances, the boiling-point of which rises constantly during the distillation, has been carefully investigated by F. D. Brown (Trans. Chem. Soc. 1879, p. 550; 1880, pp. 49 & 304; 1881, p. 517)*; and his results may be briefly stated as follows:—Calling the relative weights of the two liquids at any instant in the still W_1 and W_2, the relative weights at the same moment in the vapour (and therefore in the distillate at this instant coming over from the mixture $W_1 + W_2$) x_1 and x_2, and the vapour-pressures of the pure substances at the boiling-point of the mixture, P_1 and P_2, the composition of the instantaneous distillate is given approximately by the equation

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[*] Read November 10, 1893.

$$\frac{x_1}{x_2} = \frac{W_1}{W_2} \cdot \frac{P_1}{P_2},$$

but by substituting a constant, c, for the ratio $\frac{P_1}{P_2}$, thus

$$\frac{x_1}{x_2} = c \cdot \frac{W_1}{W_2},$$

a still better result is obtained.

The ratio $\frac{P_1}{P_2}$ would, of course, vary to some extent during the distillation; the value of the constant, c, does not differ greatly from the mean value of the ratio $\frac{P_1}{P_2}$.

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* For an account of the experimental work and of the theoretical conclusions relating to the distillation of pairs of liquid that are (*a*) non-miscible, (*b*) miscible within limits, (*c*) miscible in all proportions, the article by one of us on "Distillation" in Thorpe's 'Dictionary of Applied Chemistry' may be consulted. Pairs of liquids belonging to the first and second classes boil at a lower temperature than even the more volatile component when distilled alone, and no separation can be effected by fractional distillation. Among those in the third class there are some from which only one of the two substances can be so separated; in each case of this kind there is a particular mixture (the composition of which varies to some extent with the pressure) which distils at a constant temperature without change of composition, and it is this mixture that would be separated by fractional distillation from that one of the pure substances which is present in excess. In some cases—such as propyl alcohol and water—the mixture of constant boiling-point and composition boils at a lower temperature than either of the liquids when distilled alone; in other cases—for example, formic acid and water—this mixture has a higher boiling-point than either component.

It is only when the boiling-point of every possible mixture lies between those of the components that both liquids can be separated by fractional distillation; it is only to such cases that Brown's law is applicable, and, in the case of three liquids, it is only such mixtures that are considered in this paper.

This equation of Brown's may be written in the form

$$\frac{d\xi}{d\eta} = c\,\frac{\xi}{\eta},$$

where ξ = residue of liquid A at any instant,

" η = " B "

Taking L and M as the weights of A and B originally present, and $L + M = 1$, we obtain by integration

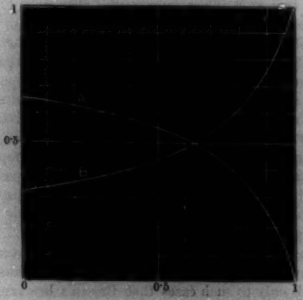
$$\frac{M}{L} y\{c+(1-c)y\}^{c-1} = c^c (1-x)^{c-1} (1-y)^c,$$

where y = quantity of the more volatile liquid A in unit weight of the distillate coming over at the instant when x is the quantity of liquid distilled.

By means of this equation we may trace the changes of composition that take place in the course of a distillation, and the variation in the composition of the distillate may be represented graphically.

To take a very simple case, let us suppose that $c = 2$, and that $L = M = \frac{1}{2}$.

Fig. 1.



In the diagram (fig. 1) the amounts of distillate that have been collected are represented as abscissæ, and the relative quantities of the two liquids A and B in the distillate at any moment as ordinates.

It will be seen that the composition of the distillate alters slowly at first, then more and more rapidly, also that while the first portion of the distillate contains a considerable amount of the less volatile substance B, the last portion is very nearly free from the lower-boiling component A. These points are fully confirmed by experiment.

By fractionating a few times in the ordinary way, collecting the distillates in six or eight fractions, we shall have a very large excess of A in the first fraction, and of B in the last.

Suppose now that we have two of these fractions, one containing A and B in the ratio of 9 : 1, and the other in the ratio of 1 : 9, and that we distil these fractions separately and completely; the results will then be represented by figs. 2 and 3.

Fig. 2.

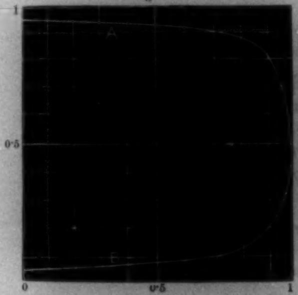
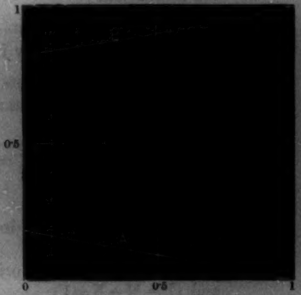


Fig. 3.



In making use of this formula it is assumed that no condensation (and therefore no fractionation) goes on in the still-head, but that the vapour reaches the condenser in the same state as when first evolved from the liquid in the still. It is obvious that by using a long still-head or a dephlegmator a more rapid separation would be effected.

In any case it is evident that after a sufficient number of fractionations, the first portion of the distillate from the first fraction will be free from B, while the residue from the last fraction will be free from A.

Suppose now that our mixture contains a third substance, C, the boiling-point of which is higher than that of B.

It may be conjectured that in the progress of the distillation, the composition of the distillate at any instant will be analogous to that determined experimentally by Brown in the case of two liquids: namely, "the proportion of the three substances in the vapour forming the instantaneous distillate is the same as that of the weights of the three substances in the residue in the still, each weight being multiplied by a suitable constant which is roughly proportional to the vapour-pressure of the corresponding liquid."

With this assumption it is easy, as in the case of two liquids, to calculate curves representing graphically the progress of the distillation.

Let ξ, η, ζ = weights of the three liquids in the still at any instant;

$\therefore d\xi, d\eta, d\zeta$ = weights of the three substances in the instantaneous distillate.

By hypothesis,

$$\frac{d\xi}{a\xi} = \frac{d\eta}{b\eta} = \frac{d\zeta}{c\zeta}.$$

Integrate

$$\frac{1}{a}\log\xi = \frac{1}{b}\log\eta + k = \frac{1}{c}\log\zeta + k'.$$

Let L, M, N be the original weights of the three liquids, and $(L+M+N=1)$;

$$\therefore \frac{1}{a}\log L = \frac{1}{b}\log M + k = \frac{1}{c}\log N + k'.$$

Subtract

$$\frac{1}{a}\log\frac{\xi}{L} = \frac{1}{b}\log\frac{\eta}{M} = \frac{1}{c}\log\frac{\zeta}{N};$$

$$\therefore \left(\frac{\xi}{L}\right)^{\frac{1}{a}} = \left(\frac{\eta}{M}\right)^{\frac{1}{b}} = \left(\frac{\zeta}{N}\right)^{\frac{1}{c}}.$$

Let

$$\left.\begin{array}{l} y_1 = \text{weight of A} \\ y_2 = \quad ,, \quad \text{B} \\ y_3 = \quad ,, \quad \text{C} \end{array}\right\} \begin{array}{l} \text{in unit weight of distillate} \\ \text{coming over when } x = \text{weight} \\ \text{of liquid distilled;} \end{array}$$

$$\therefore \quad y_1 = \frac{d\xi}{d\xi + d\eta + d\zeta} = \frac{a\xi}{a\xi + b\eta + c\zeta};$$

$$y_2 = \frac{b\eta}{a\xi + b\eta + c\zeta}; \quad y_3 = \frac{c\zeta}{a\xi + b\eta + c\zeta}.$$

Let

$$\frac{\xi}{\mathrm{N}} = z, \quad \frac{a}{c} = \lambda + 1, \quad \frac{b}{c} = \mu + 1;$$

$$\therefore \quad \xi = \mathrm{L}z^{\lambda+1}, \quad \eta = \mathrm{M}z^{\mu+1}, \quad \zeta = \mathrm{N}z;$$

$$\therefore \quad 1 - x = \xi + \eta + \zeta = \mathrm{L}z^{\lambda+1} + \mathrm{M}z^{\mu+1} + \mathrm{N}z;$$

$$y_1 = \frac{a\mathrm{L}z^{\lambda}}{a\mathrm{L}z^{\lambda} + b\mathrm{M}z^{\mu} + c\mathrm{N}}; \quad y_2 = \frac{b\mathrm{M}z^{\mu}}{a\mathrm{L}z^{\lambda} + b\mathrm{M}z^{\mu} + c\mathrm{N}};$$

$$y_3 = \frac{c\mathrm{N}}{a\mathrm{L}z^{\lambda} + b\mathrm{M}z^{\mu} + c\mathrm{N}}.$$

The elimination of z from the expressions for x and y is impracticable, but the curves may be readily traced by treating z as an independent variable.

In the following curves (figs. 4 & 5) we have taken

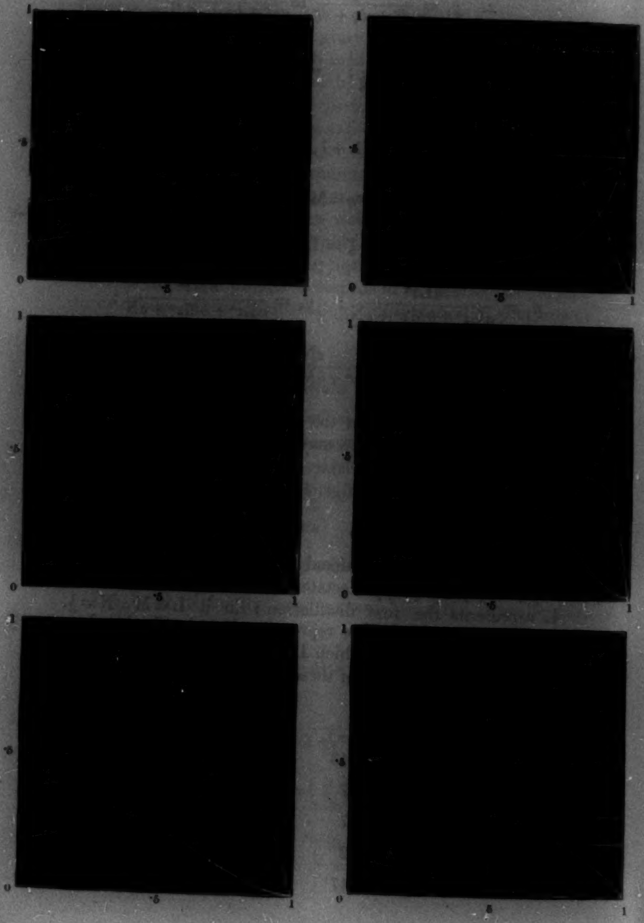
$$a=4, \quad b=2, \quad c=1, \quad [\text{and } \therefore \lambda=3, \ \mu=1],$$

these being nearly proportional to the vapour-pressures of methyl, ethyl, and propyl acetates.

I. represents the first distillation; in it $\mathrm{L}=\mathrm{M}=\mathrm{N}=\frac{1}{3}$. II. α, II. β, II. γ, II. δ, II. ϵ represent what would take place if the five fractions into which I. is divided were separately distilled; the composition of these fractions being found from the curves I. to be:—

	A or L.	B or M.	C or N.
II. α	·543	·300	·157
II. β	·47	·33	·20
II. γ	·37	·365	·265
II. δ	·22	·39	·39
II. ϵ	·047	·265	·687

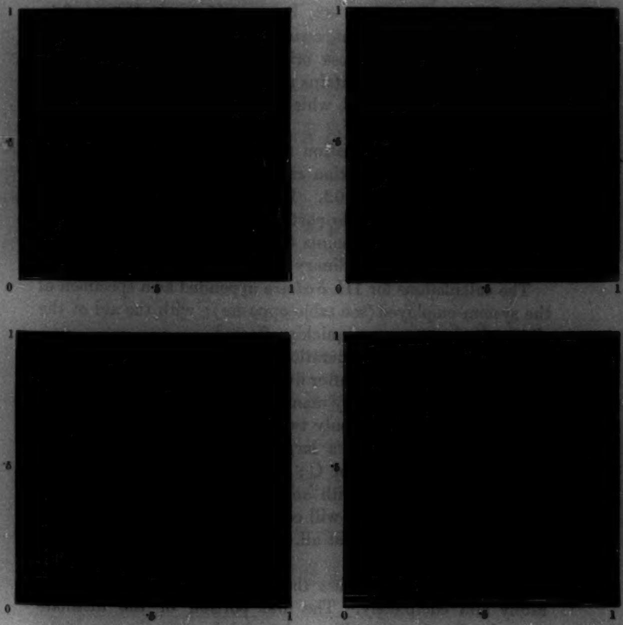
Fig. 4.



It is to be noted that in II. *α*, which is richest in the low-boiling liquid A, the amount of A rises from ·543 to an initial value ·74, and in the first half (or at least ⅖) of the distillate the aggregate amount of A rises to ·7.

In II. *ε*, which is richest in the high-boiling liquid C, the amount of C rises from ·687 to about ·9 in the last fifth, or ·82 in the last two-fifths of the distillate; whereas in II. *δ*, which is richest in B, the middle liquid, the amount of B rises merely from ·39, its initial value, to about ·45, its value

Fig. 5.



for the fifth which is collected just after the first half (·5 to ·7). It is curious that in each of these distillations B rises to practically the same maximum value, viz. ·45, the effect of the varying richness of the fractions in regard to B being merely to widen out or contract the B curve.

III. α *a*, III. δ "*c d*," III. ε *e* represent the distillation of those fractions of II. which are richest in A, B, and C respectively, and give some indication of the processes going on at the important stages of the third fractionation. In these A rises from ·7 to ·82, B from ·45 to ·5, C from ·9 to ·98. Here we notice that, owing to the small amount of C present in III. α*a* (whose original composition is A ·696, B ·239, and C ·065), B attains a higher maximum (·525) than it does (·496) in III. δ *cd*, which is far richer in B originally [A ·182, B ·44, C ·378].

P represents a distillation at an advanced stage of the fractionations—of a fraction rich in B. Its initial composition is A ·02, B ·96, C ·02. B is only advanced from ·96 to ·97 at the most favourable part of the curve, and is nowhere very free from C; this points to the difficulty of obtaining B in a pure state by the ordinary method of fractionation.

The calculations for III. δ *cd* are appended as a specimen of the system employed (see table opposite): with the aid of the slide-rule they are very quickly effected.

Apart from the consideration of the above hypothesis and formulæ, it is clear that after five or six fractional distillations, carried out in the ordinary manner (but with a larger number of fractions than when only two substances are present), the first fraction will contain a large excess of A, a little of B, but very little, if any, of C; the middle fraction will contain an excess of B with moderate quantities of both A and C; the last fraction will consist almost entirely of C, the quantity of A, if present at all, being much smaller than that of B.

Now let us suppose that these fractions are distilled separately and completely. The first portion of the distillate from the first fraction will consist of A in a nearly pure state, and, even if it contains a little of B, will be free from C. Of

L = ·182. $\quad 1-x = \cdot 182\, z^4 + \cdot 44\, z^2 + \cdot 378\, z$;

M = ·44.

N = ·378. $\quad y_3 = \dfrac{378}{378 + 880\, z + 728\, z^3}$.

z.	·378 z.	·44 z^2.	·182 z^4.	$1-x$.	880 z.	728 z^3.	Denominator.	y_3.	y_2.	y_1.
0	0	0	0	0	0	0	378	1	0	0
·1	·038	·004	·000	·042	88	1	467	·810	·188	·002
·2	·075	·018	·000	·093	176	6	560	·675	·314	·011
·3	·114	·040	·001	·155	264	20	662	·571	·399	·030
·4	·151	·070	·005	·226	352	46	776	·488	·453	·059
·5	·189	·110	·011	·310	440	91	909	·416	·484	·100
·6	·227	·158	·024	·409	528	157	1063	·356	·496	·148
·7	·265	·216	·044	·525	616	249	1243	·304	·495	·200
·8	·303	·282	·074	·659	704	373	1455	·260	·483	·256
·9	·340	·356	·119	·815	792	530	1700	·222	·466	·312
1·0	·378	·440	·182	1·000	880	728	1986	·190	·443	·367

the middle fraction the middle portion (or probably a rather later one) will be purest, but will contain a little of both A and C. The purest part of the last fraction will be the last portion (the residue in the still) and will be quite free from A though it may contain a little of B.

By proceeding with the fractionation in the usual manner we could obtain A and C in a pure state by separating each time the first small portion of the first distillate and the residue from the last distillate : but we should have to carry the fractionation a very great deal farther before we could obtain B in a state of purity. We should, in fact, have to reduce the quantities of A and C in this fraction until they both disappeared from the middle portion of the distillate, and an idea of the difficulty of attaining such a result may be gathered from the diagram P.

With substances whose boiling-points are not very far apart the rise of temperature during such a distillation would be almost imperceptible, and in the fractionation it would be necessary to collect a large number of fractions above and below the true boiling-point of B with very small temperature-ranges. It would be necessary also to read the temperature with extreme care, and to make very accurate corrections for changes of the barometer (which would be impossible unless the vapour-pressures of the substance had previously been determined).

It has been pointed out, however, that after the mixture has been fractionated a few times, the first portion of the first distillate is free from C, while the residue from the last distillate is free from A. If, therefore, after the fifth or sixth fractionation we remove these first and last portions each time, we shall gradually accumulate two quantities of liquid, one (the first portions) containing only A and B, the other (the residues) only B and C, and these may be separately fractionated in the ordinary way.

It is to be noticed also that as a quantity of liquid is removed each time, the total amount to be distilled, and therefore the time required for the distillation, gradually diminishes.

In arranging the fractions—as regards temperature—it is advisable, if the true boiling-points of the liquids are known, to take these temperatures and also the middle temperatures between the boiling-points of A and B and of B and C as definite points, and to arrange the fractions as far as possible symmetrically about these points. There will thus be fractions of small and diminishing temperature-range above and below the boiling-point of B—and at the boiling-points of A and C after a few fractionations and until these substances are to a large extent eliminated—and fractions of large and increasing temperature-range above and below the two middle temperatures.

As the preliminary fractionation proceeds, more and more of A and C will be eliminated, so that after a time the boiling-point of the first portion of the lowest fraction will gradually rise, while that of the residue from the last fraction will gradually fall until either A or C disappear, or, if we proceed still further, until we have nothing but B left. Much time may therefore be saved if, instead of adding all the first portions of the lowest fractions together (and the same remark applies to the residues from the last fractions), we collect them in six or more different quantities. The final fractionations will then be in a forward state when the preliminary one is completed.

We shall in this manner obtain two quantities of B, one by the final fractionation of the first portions of the distillates from the lowest fractions of the preliminary fractionations, the other by that of the residues from the preliminary fractionations, and we can test the purity of the substance by comparing the boiling-points and the specific gravities of the two samples so obtained.

In order to test the value of this method, a mixture of 200 cub. cent. of methyl acetate (B.P. 57·1), 250 cub. cent. of ethyl acetate (B.P. 77·15), and 200 cub. cent. of propyl acetate* (B.P. 101·55) was distilled from a flask provided with a still-head one metre in length. The distillate was collected in four approximately equal fractions, numbered 5, 8, 11, and 14 under F in the table (I.) below. The observed temperatures (reduced to 760 millim.), the range of temperature for each fraction (Δt), the weight of each fraction (Δw), and the ratio of the weight to the temperature-range $\left(\frac{\Delta w}{\Delta t}\right)$ are also given in the tables. The second fractionation (II.) was carried out in the following manner:—the first fraction from I. (No. 5, B. P. 63°·8 to 71°·3) was distilled and the distillate collected in a separate receiver (No. 4) until the temperature rose to 63°·8 *, when receiver No. 5 was substituted for it and the distillation continued until the temperature rose to 71°·0. The gas was then turned out, and the second fraction from I. (No. 8, B. P. 71°·3 to 77°·8) added to the residue in the flask. Heat was again applied and the distillate collected in receiver No. 5 until the temperature again rose to 71°·0, when it was replaced by receiver No. 8, and the distillation was continued until the temperature rose to 77°·1. The third fraction from I. (No. 11, B. P. 77°·8 to 89°·2) was then added to the residue in the flask, and the distillate collected in No. 8 until the temperature again rose to 77°·1, when No. 11 was put in its place. When the temperature had risen to 84°·4 a new receiver, No. 13, was substituted for No. 11, and the distillation was continued until the thermometer registered 91°·7. The last fraction from I. (No. 14, B.P. above 89°·2) was then added to the residue in the flask and the distillate collected in No. 13 until the temperature rose again to 91°·7, after which fractions were collected in No. 14 from 91°·7 to 98°·4, and No. 15 from 98°·4 to 101°·5, when the distillation was stopped, and the residue poured into receiver No. 17.

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* The esters had previously been purified with great care, and were distilled separately from phosphorus pentoxide immediately before being mixed. Special precautions were also taken to avoid absorption of moisture during the fractionations. The methods of preparation and purification and proofs of the purity of the esters are fully described in the Trans. Chem. Soc. lxiii. p. 1194.

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* The actual temperature was 63°·5, the barometric pressure being 751·1 millim.; the value of $\frac{dp}{dt}$ at the boiling-point is 27 millim. per degree for methyl acetate, 25 millim. for ethyl acetate, and 23 millim. for propyl acetate. All the temperatures are corrected to 760 millim. by means of these constants.

I.

F.	t corr. to 760 mm.	Δt.	Δw.	Δw./Δt
5.	63°·8–71°·3	7°·5	141·4	18·9
8.	71·3–77·8	6·5	145·2	22·3
11.	77·8–89·2	11·4	146·9	12·8
14.	above 89·2	......	143·3	
			576·8	

II.

F.	t corr. to 760 mm.	Δt.	Δw.	Δw./Δt
4.	60°·9– 63°·8	2°·9	59·7	20·6
5.	63·8– 71·0	7·2	134·2	18·6
8.	71·0– 77·1	6·1	98·4	16·1
11.	77·1– 84·4	7·3	67·2	9·2
13.	84·4– 91·7	7·3	51·0	7·0
14.	91·7– 98·4	6·7	86·1	12·7
15.	98·4–101·5	3·1	63·7	20·6
17.	above 101·5	......	14·3	
			574·6	

III.

F.	t corr. to 760 mm.	Δt.	Δw.	Δw./Δt
3.	58°·4 – 60°·95	2°·55	36·5	14·3
4.	60·95– 63·85	2·9	77·2	26·6
5.	63·85– 71·0	7·15	100·6	14·0
8.	71·0 – 77·15	6·15	75·4	12·4
11.	77·15– 84·4	7·25	67·7	9·3
13.	84·4 – 91·65	7·25	59·6	8·2
14.	91·65– 98·4	6·75	46·7	6·9
15.	98·4 –101·45	3·05	78·3	26·1
17.	101·45–101·55	0·1	19·7	197·0
Z ...	....	......	10·0	
			571·7	

IV.

F.	t corr. to 760 mm.	Δt.	Δw.	Δw./Δt
2.	57°·7 – 58°·6	0°·9	17·2	19·1
3.	58·6 – 61·0	2·4	58·1	23·7
4.	61·0 – 63·9	2·9	53·1	18·3
5.	63·9 – 71·05	7·15	85·1	11·8
8.	71·05– 77·15	6·1	66·6	10·9
11.	77·15– 84·3	7·15	84·7	11·8
13.	84·3 – 91·6	7·3	40·8	5·6
14.	91·6 – 98·4	6·8	33·8	5·0
15.	98·4 –100·95	2·5	44·1	17·6
16.	100·95–101·45	0·5	38·7	77·4
17.	101·45–101·55	0·1	24·2	242·0
Z ...	....	......	20·2	
			566·6	

V.

F.	t corr. to 760 mm.	Δt.	Δw.	Δw./Δt
2.	57°·1 – 58°·2	1°·1	19·4	17·6
3.	58·2 – 60·7	2·5	69·9	28·0
4.	60·7 – 63·85	3·15	49·0	15·3
5.	63·85– 71·0	7·15	69·0	9·6
8.	71·0 – 77·15	6·15	71·1	11·6
11.	77·15– 83·35	6·2	85·1	13·6
13.	83·35– 90·75	7·4	33·7	4·6
14.	90·75– 98·45	7·7	33·5	4·4
15.	98·45–100·95	2·5	29·4	11·8
16.	100·95–101·45	0·5	36·7	73·4
17.	101·45–101·55	0·1	31·1	311·0
Z ...	....	......	34·7	
			562·6	

VI.

F.	t corr. to 760 mm.	Δt.	Δw.	Δw./Δt
1.	57°·0 – 57°·3	0°·3	6·3	21·0
2.	57·3 – 58·2	0·9	28·5	31·7
3.	58·2 – 60·1	1·9	55·0	29·0
4.	60·1 – 63·85	3·75	54·2	14·3
5.	63·85– 70·75	6·9	47·4	6·9
7.	70·75– 75·7	4·95	52·0	10·4
8.	75·7 – 77·15	1·45	33·4	22·3
11.	77·15– 78·4	1·25	34·8	29·0
12.	78·4 – 82·5	4·1	47·9	11·7
13.	82·5 – 90·2	7·7	28·1	3·6
14.	90·2 – 98·45	8·25	27·8	3·3
15.	98·45–100·95	2·5	25·6	10·2
16.	100·95–101·45	0·5	30·3	60·6
17.	101·45–101·55	0·1	34·0	340·0
Z ...	....	......	50·6	
			555·9	

VII.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
1.	57°·05 – 57°·25	0°·2	13·3	66·5
2.	57·25 – 58·15	0·9	42·9	47·7
3.	58·15 – 60·0	1·85	47·5	25·0
4.	60·0 – 63·8	3·8	43·5	11·4
5.	63·8 – 70·75	6·95	45·9	6·7
7.	70·75 – 75·8	5·05	42·7	8·4
8.	75·8 – 77·2	1·4	45·0	32·1
11.	77·2 – 78·3	1·1	37·8	34·4
12.	78·3 – 82·1	3·8	41·1	10·8
13.	82·1 – 90·15	8·05	23·4	2·9
14.	90·15 – 98·45	8·3	22·3	2·7
15.	98·45 – 100·95	2·5	17·6	7·0
16.	100·95 – 101·45	0·5	24·3	48·6
17.	101·45 – 101·55	0·1	23·7	237·0
Z ...	....	......	80·2	
			551·2	

VIII.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
A ...	° °	°	6·0	
1.	57·15 – 57·35	0·2	11·7	58·5
2.	57·35 – 58·25	0·9	49·1	54·6
3.	58·25 – 60·15	1·9	45·4	23·9
4.	60·15 – 63·8	3·65	36·0	9·5
5.	63·8 – 70·7	6·9	37·8	5·5
7.	70·7 – 75·8	5·1	41·9	8·2
8.	75·8 – 77·15	1·35	43·8	32·4
11.	77·15 – 78·2	1·05	51·1	48·7
12.	78·2 – 81·1	2·9	34·1	11·8
13.	81·1 – 90·05	8·95	22·8	2·5
14.	90·05 – 98·5	8·45	17·8	2·1
15.	98·5 – 101·0	2·5	16·7	6·7
16.	101·0 – 101·45	0·45	19·4	43·3
17.	101·45 – 101·55	0·1	12·6	126·0
Z ...	....	......	101·5	
			547·7	

IX.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
A ...	° °	°	11·4	
1.	57·0 – 57·25	0·25	18·6	74·4
2.	57·25 – 58·0	0·75	44·5	59·3
3.	58·0 – 59·75	1·75	41·4	23·0
4.	59·75 – 63·75	4·0	35·1	8·8
5.	63·75 – 71·0	7·25	30·4	4·2
7.	71·0 – 75·95	4·95	39·9	8·0
8.	75·95 – 77·15	1·2	47·9	39·9
11.	77·15 – 78·05	0·9	57·9	64·3
12.	78·05 – 80·55	2·5	27·1	10·8
13.	80·55 – 89·9	9·35	20·4	2·2
14.	89·9 – 98·85	8·95	14·1	1·6
15.	98·85 – 101·05	2·2	12·8	5·8
16.	101·05 – 101·45	0·4	9·8	24·5
17.	101·45 – 101·55	0·1	6·3	63·0
Z ...	....	......	124·6	
			542·2	

X.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
A ...	° °	°	21·5	
1.	57·1 – 57·25	0·15	16·7	111·3
2.	57·25 – 57·75	0·5	35·7	64·9
3.	57·75 – 59·35	1·6	43·0	26·9
4.	59·35 – 63·75	4·4	34·9	7·9
5.	63·75 – 71·3	7·55	28·9	3·8
7.	71·3 – 76·1	4·8	36·9	7·7
8.	76·1 – 77·15	1·05	57·3	54·6
11.	77·15 – 77·9	0·75	53·9	71·9
12.	77·9 – 79·9	2·0	23·3	11·7
13.	79·9 – 89·9	10·0	19·8	2·0
14.	89·9 – 99·3	9·4	12·5	1·3
15.	99·3 – 101·1	1·8	9·7	5·4
16.	101·1 – 101·5	0·4	8·4	21·0
17.	101·5 – 101·55	0·05	3·5	70·0
Z ...	....	......	132·6	
			538·6	

XI.

F.	t. corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w.}{\Delta t}$
	° °	°		
A ...	....	......	31·5	
1.	57·1 – 57·2	0·1	15·2	152·0
2.	57·2 – 57·7	0·5	34·6	69·2
3.	57·7 – 59·1	1·4	36·4	26·0
4.	59·1 – 63·4	4·3	34·1	7·9
5.	63·4 – 71·5	8·1	23·9	3·0
7.	71·5 – 76 25	4·75	36·1	7·5
8.	76·25– 76·95	0·7	28·5	40·7
9.	76·95– 77·15	0·2	27·9	139·5
10.	77·15– 77·35	0·2	28·7	143·5
11.	77·35– 77·75	0 4	29·4	73·5
12.	77·75– 79·35	1·6	21·9	13·7
13.	79·35– 89·3	9·95	18·7	1·9
14.	89·3 – 99·55	10·25	10·9	1·1
15.	99·55–101·15	1·6	9·5	5·9
16.	101·15–101·45	0·3	6·0	20·0
Y ...	....	......	8·1	
Z ...	....	......	132·6	
			534·0	

XII.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w.}{\Delta t}$
	° °	°		
A ...	....	......	41·1	
1.	57·1 – 57·2	0·1	17·0	170·0
2.	57·2 – 57·65	0·45	30·5	67·8
3.	57·65– 58·9	1·25	30·3	24·2
4.	58·9 – 63·5	4·6	31·0	6 7
5.	63·5 – 71·9	8·4	23·8	2·8
7.	71·9 – 76·45	4·55	32·9	7·1
8.	76·45– 77·0	0·55	30·5	55·5
9.	77·0 – 77·15	0·15	27·6	184·0
10.	77·15– 77·3	0·15	36·2	241·3
11.	77·3 – 77·6	0·3	25·6	85·3
12.	77·6 – 78·85	1·25	18·7	14·9
13.	78·85– 88·9	10·05	17·9	1·8
14.	88·9 – 99·55	10·65	9·5	0·9
15.	99·55–101·15	1·6	8·1	5·1
X ...	....	......	7·8	
(Y & Z.) }	....	......	140·7	
			529·2	

XIII.

F.	t. corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w.}{\Delta t}$
	° °	°		
A ...	....	......	50·4	
1.	57·1 – 57·2	0·1	20·4	204·0
2.	57·2 – 57·6	0·4	22·9	57·2
3.	57·6 – 58·7	1·1	26·0	23·6
4.	58·7 – 63·8	5·1	30·8	6·0
5.	63·8 – 72·35	8·55	21·3	2·5
7.	72·35– 76·65	4·3	35·2	8·2
8.	76·65– 77·05	0·4	23·2	58·0
9.	77·05– 77·15	0·1	33·1	331·0
10.	77·15– 77·25	0·1	38·0	380·0
11.	77·25– 77·5	0·25	21·5	86·0
12.	77·5 – 78·5	1·0	17·6	17·6
13.	78·5 – 88·7	10·2	18·7	1·8
14.	88·7 – 99·7	11 0	9 3	0·8
W ...	....	......	8·1	
(X to Z.) }	....	......	148·5	
			525·0	

XIV.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w.}{\Delta t}$
	° °	°		
A ...	....	......	60·7	
1.	57·1 –57·2	0·1	14·7	147·0
2.	57·2 –57·6	0·4	23·6	59·0
3.	57·6 –58·6	1·0	22·6	22·6
4.	58·6 –64·25	5·65	29·9	5·3
5.	64·25–72·8	8·55	18·8	2·2
7.	72·8 –76·65	3·85	28·8	7·5
8.	76·65–77·05	0·4	27·1	67·7
9.	77·05–77·15	0·1	35·2	352 0
10.	77·15–77·25	0·1	43·1	431·0
11.	77·25–77·45	0·2	15·6	78·0
12.	77·45–78·2	0·75	17·4	23·2
13.	78·2 –84·7	6·5	16·0	2·4
V ...	....	......	11·4	
(W to Z.) }	....	......	156·6	
			521·5	

XV.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
	° °	°		
A ...	....	......	70·1	
1.	57·1 –57·2	0·1	16·3	163·0
2.	57·2 –57·6	0·1	21·0	52·5
3.	57 6 –58·6	1·0	17·1	17·1
4.	58·6 –64·8	6·2	27·1	4·4
5.	64·8 –73·4	8·6	17·4	2·0
7.	73·4 –76·7	3·3	27·8	8·4
8.	76·7 –77·05	0·35	21·7	62·0
9.	77·05–77·15	0·1	43·0	430·0
10.	77·15–77·2	0·05	40·4	808·0
11.	77·2 –77·4	0·2	21·2	106·0
12.	77·4 –78·0	0·6	12·5	20·8
13.	78·0 –79·6	1·6	9·7	6·1
V ...	....	......	15·8	
(W to Z.)	....	......	156·6	
			517·7	

XVI.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
	° °	°		
A ...	....	......	79·9	
1.	57·1 –57·2	0·1	14·6	146·0
2.	57·2 –57·6	0·4	17·3	43·2
3.	57·6 –58·6	1·0	14·6	14·6
4.	58·6 –65·35	6·75	25·0	3·7
5.	65·35–73·9	8·55	15·6	1·8
7.	73·9 –76·8	2·8	26·6	9·2
8.	76·8 –77·1	0·3	26·7	89·0
9.	77·1 –77·15	0·05	39·0	780·0
10.	77·15–77·2	0·05	39·9	798·0
11.	77·2 –77·35	0·15	24·4	162·6
12.	77·35–77·9	0·55	11·6	21·1
U ...	....	......	6·6	
(V to Z.)	....	......	172·4	
			514·2	

XVII.

	° °	°		
A ...	....	......	86·7	
1.	57·1 –57·2	0·1	14·4	144·0
3.	57·2 –57·6	0·4	14·3	35·7
3.	57·6 –58·6	1·0	13·5	13·5
4.	58·6 –65·8	7·2	22·4	3·1
5.	65·8 –74·4	8·6	15·9	1·8
7.	74·4 –76·85	2·45	24·7	10·1
8.	76·85–77·1	0·25	19·0	76·0
9.	77·1 –77·15	0·05	38·3	766·0
10.	77·15–77·2	0·05	48·0	960·0
11.	77·2 –77·3	0·1	22·0	220·0
12.	77·3 –77·65	0·35	9·6	27·5
U ...	....	......	10·2	
(V to Z.)	....	......	172·4	
			511·4	

XVIII.

	° °	°		
A ...	....	......	94·1	
1.	57·1 –57·2	0·1	11·3	113·0
2.	57·2 –57·6	0·4	13·4	33·5
3.	57·6 –58·6	1·0	10·7	10·7
4.	58·6 –66·25	7·65	21·3	2·8
5.	66·25–74·85	8·7	15·0	1·7
7.	74·85–76·85	2·0	21·4	10·7
8.	76·85–77·1	0·25	20·5	82·0
9.	77·1 –77·15	0·05	41·2	814·0
10.	77·15–77·2	0·05	49·6	992·0
11.	77·2 –77·3	0·1	21·0	210·0
T ...	....	......	6·0	
(U to Z.)	....	......	182·6	
			508·1	

XIX.

	° °	°		
A ...	....	......	100·3	
1.	57·1 –57·2	0·1	9·5	95·0
2.	57·2 –57·6	0·4	11·7	29·2
3.	57·6 –58·6	1·0	9·0	9·0
4.	58·6 –66·75	8·15	19·0	2·3
5.	66·75–75·35	8·6	15·7	1·8
7.	75·35–76·85	1·5	18·4	12·3
8.	76·85–77·1	0·25	18·3	73·2
9.	77·1 –77·15	0·05	39·5	790·0
10.	77·15–77·2	0·05	57·6	1152·0
11.	77·2 –77·3	0·1	13·9	139·0
T ...	....	......	9·7	
(U to Z.)	....	......	182·6	
			505·2	

XX.

	° °	°		
A ...	....	......	105·5	
1.	57·1 –57·15	0·05	5·9	118·0
2.	57·15–57·55	0·4	10·7	26·7
3.	57·55–58·55	1·0	8·9	8·9
4.	58·55–67·25	8·7	17·6	2·0
5.	67·25–75·95	8·7	16·6	1·9
7.	75·95–76·95	1·0	16·0	16·0
8.	76·95–77·15	0·2	13·4	67·0
9.	77·15	0·05	45·0	1888·0
10.	to 77·2		49·4	
11.	77·2 –77·25	0·05	13·0	260·0
S ...	....	......	8·6	
(T to Z.)	....	......	192·3	
			502·9	

XXI.

F.	*t* corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
	° °	°		
A ...	....	......	105·5	
B ...	–57·15	......	11·6	
2.	57·15–57·55	0·4	7·2	18·0
3.	57·55–58·55	1·0	7·0	7·0
4.	58·55–67·2	8·65	16·3	1·8
5.	67·2 –76·3	9·1	18·2	2·0
7.	76·3 –77·0	0·7	11·2	16·0
8.	77·0 –77·15	0·15	25·3	168·6
9.	77·15 slight rise	0·0	38·7	very high
10.			35·5	
11.	77·15–77·2	0·05	14·8	296·0
S ...	....	......	16·8	
(T to Z.)	....	......	192·3	
			500·4	

XXII.

F.	*t* corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
	° °	°		
(A & B.)	....	......	117·1	
C ...	–57·55	......	10·1	
3.	57·55–58·55	1·0	6·0	6·0
4.	58·55–67·2	8·65	12·6	1·5
5.	67·2 –76·5	9·3	19·2	2·1
7.	76·5 –77·05	0·55	15·4	28·0
8.	77·05–77·15	0·1	21·5	215·0
9.	77·15 very slight rise.	0	24·0	very high
10.			36·6	
11.	77·15	...	17·3	
R ...	....	......	9·1	
(S to Z.)	....	......	209·1	
			498·0	

XXIII.

	° °	°		
(A to C.)	....	......	127·2	
D ...	–58·55	......	8·6	
4.	58·55–67·2	8·65	9·1	1·1
5.	67·2 –76·7	9·4	21·7	2·3
7.	76·7 –77·1	0·4	14·4	36 0
8.	77·1 –77·15	0·05	16·6	332·0
9.	77·15	0	33·1	very high
10.			25 2	
11.	77·15		11·5	
R ...	....	......	19·7	
(S to Z.)	....	......	209·1	
			496·2	

XXIV.

	° °	°		
(A to C.)	....	......	127·2	
D ...	–58·55	......	10·0	
4.	58·55–67·2	8·65	6·8	0·8
5.	67·2 –75·3	8·1	15·1	1·9
6.	75·3 –76·8	1·5	7·8	5·2
7.	76·8 –77·1	0·3	12·2	40·7
8.	77·1 –77·15	0·05	14·6	292·0
9.	77·15	0	31·9	very high
10.			26·9	
Q ...	....	......	13·1	
(R to Z.)	....	......	228·8	
			494·4	

XXV.

	° °	°		
(A to D.)	....	......	137·2	
E ...	–67·9	......	10·2	
5.	67·9 –75·5	7·6	9·3	1·3
6.	75·5 –76·85	1·35	9·0	6·7
7.	76·85–77·1	0·25	13·2	52·8
8.	77·1 –77·15	0·05	20·2	404·0
9.	77·15	0	21·6	very high
10.			19·1	
Q ...	....	......	23·2	
(R to Z.)	....	......	228·8	
			492·3	

XXVI.

	° °	°		
(A to D.)	....	......	137·2	
E ...	–68·9	......	11·9	
5.	68·9 –75·7	6·8	7·1	1·0
6.	75·7 –76·9	1·2	10·4	8·7
7.	76·9 –77·1	0·2	15·4	77·0
8.	77·1 –77·15	0·05	16·4	328·0
9.	77·15	0	19·2	very high
10.			12·1	
Q ...	....	......	32·3	
(R to Z.)	....	......	228·8	
			490·8	

The third fractionation (III.) was carried out in a similar manner, a new fraction No. 3 being collected at the beginning. As the temperature rose to 101°·55, the true B.P. of propyl acetate, before the end of the last distillation, the residue in the flask consisted of this substance in a nearly pure state, and it was therefore collected in a separate flask Z, and was not redistilled during the preliminary fractionations. The residues from the subsequent fractionations up to the tenth were collected in Z, but after this, as a large amount of propyl acetate had been removed, the temperature did not reach 101°·55, and the residue from the eleventh fractionation (B. P. above 101°·45) was collected in a new flask, Y.

At the end of the twelfth fractionation the temperature rose only to 101°·15, and the residue was therefore placed in a third flask, X, and subsequently the residues were collected in W, V, &c., to Q, as shown in the tables.

It was not until the fifth fractionation that the first fraction began to boil at 57°·1, the B.P. of methyl acetate, and it was not thought advisable to separate the first portion of the first distillate until the eighth fractionation. This portion and also the corresponding ones up to the twentieth fractionation were collected together in flask A, after which the first portions were collected in B, C, D, and E.

The ratios $\frac{\Delta w}{\Delta t}$ give an indication of the purity of the various distillates, for with a pure substance, since $\Delta t=0$, $\frac{\Delta w}{\Delta t}=\infty$, while with a very impure substance there is a rapid rise of temperature during distillation, and $\frac{\Delta w}{\Delta t}$ is consequently small.

It will be seen that the ratio for the highest fraction rises rapidly, being 197 in the third fractionation and rising to 340 in the sixth, after which it diminishes again owing to removal of propyl acetate in the residues. The value for the lowest fraction increases much more slowly, reaching a maximum of 204 in the thirteenth fractionation, and afterwards diminishing owing to removal of methyl acetate.

The presence of the middle substance, ethyl acetate, is not clearly shown until the fourth fractionation, when the value of $\frac{\Delta w}{\Delta t}$ for the sixth distillate (No. 9) is higher than for those above and below it; but the gradual accumulation of the ethyl acetate in the middle fractions in subsequent fractionations is clearly shown by the rise in the values of $\frac{\Delta w}{\Delta t}$ for fractions 8 and 11, and after the tenth fractionation in fractions 9 and 10. The range of temperature for fractions 9 and 10 was gradually diminished from 0°·2 each in the eleventh fractionation until no rise could be detected; there was indeed no perceptible rise of temperature during the collection of No. 11 in the twenty-second and twenty-third fractionations. It was therefore certain that after the twenty-sixth fractionation the tenth fraction was free from propyl acetate, and that the remaining fractions Nos. 5 to 10 contained only methyl and ethyl acetates.

The preliminary series of fractionations was therefore completed, no fraction now containing more than two substances.

The progress of the separation is well seen by mapping the temperatures against the total weight of distillate collected. The curves I. to XII. represent the first twelve fractionations, the weight of distillate being given in each case as percentages. The dotted lines at the extremities of the later curves represent the methyl and propyl acetates removed in the first portions of the first distillates and in the residues.

The presence of the middle substance is clearly indicated in the fourth curve, but not in the earlier ones.

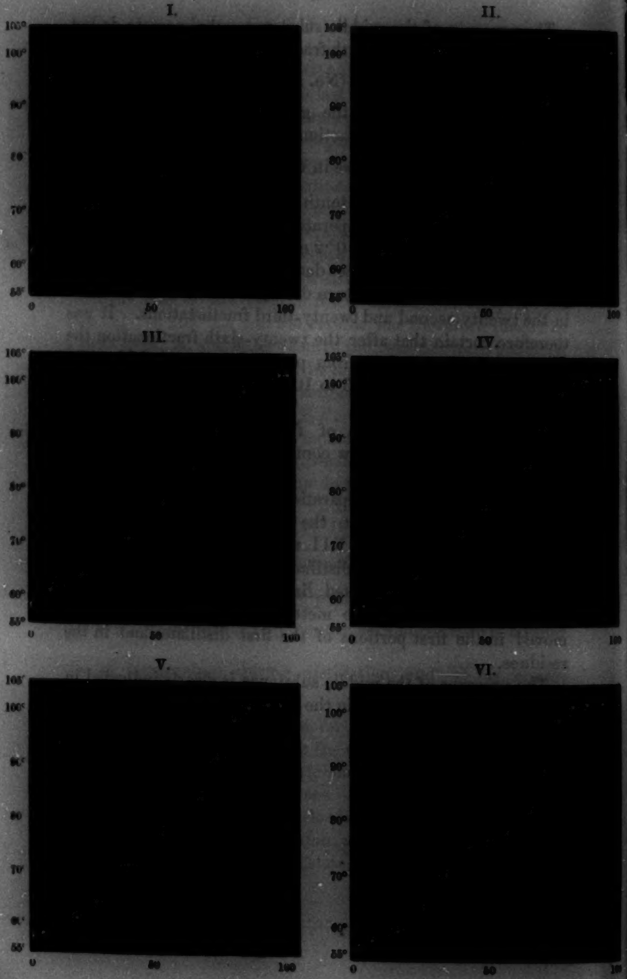
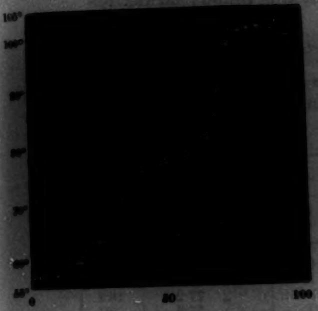
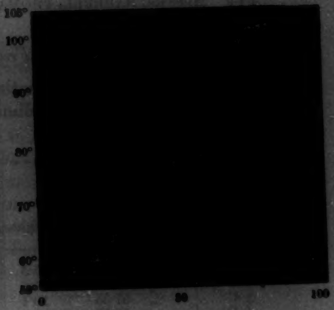


Fig. 6.

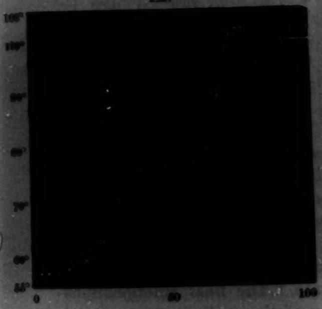
VII.



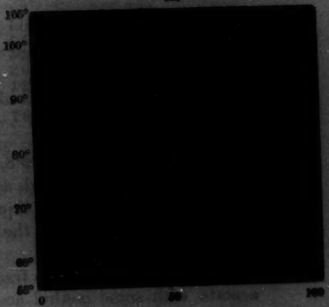
VIII.



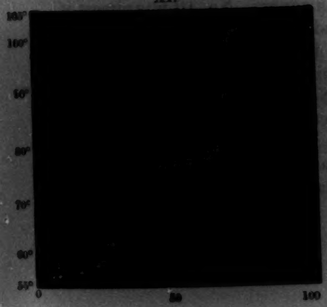
IX.



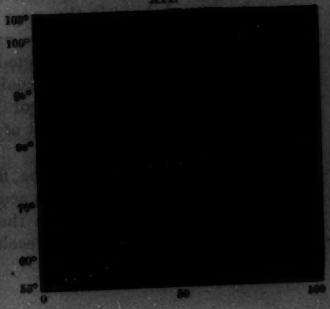
X.



XI.



XII.



Final Fractionations.

The fractions into which the esters had been separated at the end of the 26th preliminary fractionation are given below :—

Methyl and Ethyl Acetates.			Ethyl and Propyl Acetates.		
Fraction.	Temperature range.	Weight.	Fraction.	Temperature range.	Weight.
A ...	57°·1	105·5	Q ...	77°·15	32·3
B ...	below 57·15	11·6	R ...	above 77·15	19·7
C ...	„ 57·55	10·1	S......	„ 77·2	16·8
D ...	„ 58·55	10·0	T......	„ 77·3	9·7
E ...	„ 68·9	11·9	U ...	„ 77·65	10·2
5 ...	68·9–75·7	7·1	V ...	„ 79·6	15·8
6 ...	75·7–76·9	10·4	W ...	„ 99·7	8·1
7 ...	76·9–77·1	15·4	X ...	„ 101·15	7·8
8 ...	77·1–77·15	16·4	Y ...	„ 101·45	8·1
9 ...	77·15	19·2	Z ...	„ 101·55	132·6
10 ...	77·15	12·1			
		229·7			261·1

The total weight was 490·8, so that 86 grams had been lost by evaporation and through adherence to the flasks and to the funnel through which the liquid was poured into the still.

The final separation of the esters was carried out in the following manner :—

Ethyl Acetate (1).—The fractions 5 to 10 consisted of ethyl acetate with some methyl acetate. The final fractionations in this case were a continuation of the preliminary ones and were carried out in the same manner. No. 10 was almost if not quite pure ethyl acetate, and was only distilled once more; the first portion was collected in No. 9, and the residue was taken as pure. In the remaining six fractionations the residue, after the distillation of No. 9, was each time taken as pure, and the total amount of ethyl acetate thus obtained was 56·1 grams.

Ethyl Acetate (2).—The flasks Q to U contained ethyl acetate with some propyl acetate. In the six fractionations, which were carried out in the usual way, the first portion of the distillate from Q was each time taken as pure, the small residue in the still being rejected. The weight of pure ethyl acetate obtained from this series was 62·7, and the total weight was therefore 118·8 grams, or about 53 per cent. of that originally taken.

Methyl Acetate.—The flasks A to E contained methyl acetate with some ethyl acetate. Although special care had been taken to prevent absorption of moisture, it was found that a little water was present: in two fractionations, therefore, phosphorus pentoxide was placed in the distillation-bulb, which was heated by a water-bath. After the first fractionation the first portion of the distillate from the first fraction was taken as pure, the small residue in the still being each time rejected. There were altogether nine fractionations, and the amount of pure methyl acetate obtained was 88 grams, or about 48 per cent. of that taken. (The loss of this ester by evaporation would be greater than that of either of the other two.)

Propyl Acetate.—The flasks V to Z contained propyl acetate with some ethyl acetate. It was found that the first two residues were slightly acid, owing to hydrolysis of the esters by the trace of water present. These two residues were therefore collected separately and were shaken with a strong solution of potassium carbonate; the ester was then washed with water, dried with phosphorus pentoxide, and again distilled. Five fractionations were found sufficient to purify the propyl acetate, the residues being each time taken as pure (except the first two, which were treated as described). The total weight of pure propyl acetate was 126·8 grams, or 72 per cent. of that taken. (The loss by evaporation would be smallest in this case.)

The esters were then distilled from phosphorus pentoxide, and their specific gravities at 0° determined by Sprengel's method as modified by Perkin. The completeness of the separation was proved by the close agreement of the results with those obtained before mixing, as well as by the identity of the boiling-points with those previously determined.

The specific gravities are given below:—

Specific Gravities at 0°.

	Previous Determinations.	After Fractionation.
Methyl acetate * (*a*).........	·95929	·95937
„ „ (*b*).........	·95934	
Ethyl acetate * (*a*)	·92434	(1) ·92438
„ „ (*b*)	·92438	(2) ·92437
Propyl acetate * (*a*).........	·91017	·91008
„ „ (*b*).........	·91015	

* Two different samples of each ester had been purified and their specific gravities determined. The two results are given in each case, in order to show what error of experiment might be expected. The two samples of each ester were afterwards mixed together.

Apart from the comparison of the specific gravity of the ethyl acetate before mixing and after fractionation, a proof of the purity of this substance is afforded by the agreement between the specific gravities of the two samples, since, if the separation had been incomplete, one would have been contaminated with methyl acetate, the other with propyl acetate.

DISCUSSION.

Prof. Ramsay said the paper was a most valuable one, and would be a great aid to chemists. Distillations were usually carried out by "the rule of thumb," with the result that absolutely pure liquids could rarely be obtained.

The President inquired whether the curves representing the progress of distillation could be constructed from the very complete experiments made, and so test the assumed law.

Prof. Young thought this not possible from the numbers obtained. To test the law in this way would be very laborious.

XXX. *Note on the Generalizations of* Van der Waals *regarding "Corresponding" Temperatures, Pressures, and Volumes.* *By* SYDNEY YOUNG, *D.Sc., F.R.S., Professor of Chemistry, University College, Bristol*[*].

IN a paper read before the Physical Society in November 1891, and published under the above title in the Philosophical Magazine for February 1892 (vol. xxxiii. p. 153; Proc. Phys. Soc. vol. xi. p. 233), the absolute temperatures and the molecular volumes (as liquid and saturated vapour) of eleven compounds were compared with those of fluorbenzene at a series of corresponding pressures.

It was pointed out, however (p. 155), that a better mode of procedure in many respects would have been to give the temperatures, pressures, and volumes of each of the twelve substances in terms of its critical constants; but as the critical volumes of several of the compounds had not been directly determined, and as none of them were known with accuracy, the method had to be abandoned.

Since then it has been shown by M. Mathias that the critical densities—and therefore the volumes—may be determined with great accuracy by the method of Cailletet and Mathias, and as the values calculated by M. Mathias differed very slightly from those obtained independently by the same method by myself, I accepted these values as correct (Phil.

Substance.	Critical Molecular Volume.		Substance.	Critical Molecular Volume.	
	Old.	Corrected.		Old.	Corrected.
Fluorbenzene..........	270·4	270·5	Stannic chloride .	349·3	350·4
Chlorobenzene	306·5	305·7	Ether	280·7	281·4
Bromobenzene	322·4	321·4	Acetic acid.........	170·3	170·5
Iodobenzene	348·4	347·9	Methyl alcohol ...	118·0	118·0
Benzene	256·3	256·3	Ethyl alcohol ...	166·9	166·9
Carbon tetrachloride.	276·1	275·6	Propyl alcohol ...	217·6	217·6

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[*] Read November 10, 1893.

Mag. Dec. 1892). It appears, however, that there were a few small errors in the calculations, and I have therefore adopted my own values in the tables that follow. The new critical molecular volumes and those previously accepted as correct are given above; it will be seen that the alterations are very small.

The vapour-pressures, molecular volumes, and critical constants of ten esters [methyl formate, acetate, propionate, butyrate, and isobutyrate; ethyl formate, acetate, and propionate; propyl formate and acetate] have recently been determined by Mr. G. L. Thomas and myself, and the absolute temperatures, pressures, and volumes of each substance have been given in terms of its critical constants (Trans. Chem. Soc. lxiii. p. 1191).

In order to allow of a comparison being made, pressures "corresponding" with the previous ones (with a few omissions) were taken, and I now propose to give the absolute temperatures and volumes of the substances previously studied in terms of their critical constants. As the data for the esters are published in full in the Trans. Chem. Soc., it is unnecessary to reproduce them here, and it will be sufficient to give the extreme values for the ten esters in each case.

It was pointed out in the earlier paper on this subject (Phil. Mag. xxxiii. p. 172) that the compounds investigated might be arranged in groups, and this is confirmed by the method of comparison now adopted. The twenty-two compounds appear to fall into four chief groups:—

I. Benzene and its halogen derivatives, carbon tetrachloride, stannic chloride, ether.
II. The three alcohols (methyl alcohol differing, however, considerably from the others).
III. Acetic acid.
IV. The ten esters.

Ratios of Absolute Temperatures at Corresponding Pressures to Absolute Critical Temperatures.—The ratios for the compounds in groups I. are the lowest; those for the esters are mostly higher; acetic acid comes next in order; whilst the values for the alcohols are much higher than for any of the other substances.

Ratios of Boiling-Points (Abs. temps.) at Corresponding Pressures to Absolute Critical Temperatures.

Pressures C_6H_5F.	Fraction of Critical Pressure.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$	CH_3OH.	C_2H_5OH	C_3H_7OH	CH_3COOH	Ten Esters.
20	·00059	·4866	·4870	·4870	·4872	·4795	·4748	·4914	...	·5374	·5536	·5579	·5164	·4976 to ·5172
50	·00147	·5170	·5172	·5169	·5170	·5098	·5053	·5216	...	·5685	·5836	·5861	·5478	·5189 „ ·5468
100	·00295	·5431	·5429	·5427	·5427	·5359	·5313	·5471	...	·5943	·6085	·6101	·5743	·5447 „ ·5718
200	·00590	·5723	·5723	·5717	·5717	·5648	·5608	·5763	·5777	·6229	·6361	·6365	·6041	·5740 „ ·5999
400	·01180	·6054	·6052	·6049	·6049	·5989	·5943	·6089	·6097	·6550	·6665	·6656	·6372	·6070 „ ·6314
760	·02241	·6400	·6402	·6403	·6401	·6334	·6293	·6435	·6444	·6882	·6981	·6960	·6716	·6413 „ ·6650
1500	·04423	·6827	·6833	·6833	·6830	·6765	·6728	·6855	·6866	·7277	·7355	·7326	·7133	·6838 „ ·7059
3000	·08846	·7334	·7345	·7343	·7337	·7282	·7251	·7357	·7371	·7734	·7794	·7736	·7624	·7348 „ ·7541
5000	·14740	·7771	·7786	·7777	...	·7725	·7697	·7788	·7798	·8119	·8161	·8118	·8031	·7778 „ ·7949
7000	·20640	·8092	·8103	·8090	...	·8052	·8025	·8106	·8113	·8396	·8429	·8384	·8322	·8101 „ ·8248
10000	·29490	·8464	·8466	...	...	·8429	·8411	·8463	·8472	·8708	·8738	·8690	·8651	·8452 „ ·8595
15000	·44230	·8930	·8926	...	...	·8906	·8889	·8934	·8923	·9098	·9119	·9082	·9057	·8915 „ ·9028
20000	·58990	·9288	·9285	...	...	·9270	·9260	·9287	·9277	·9381	·9412	·9383	·9368	·9279 „ ·9357
25000	·73720	·9579	·9576	...	...	·9566	·9565	·9577	·9573	·9643	·9652	·9637	·9626	·9574 „ 9624
28000	·82570	·9731	·9727	...	...	·9725	·9725	·9732	·9728	·9775	·9775	·9771	·9740	·9729 „ ·9762
30000	·88460	·9829	·9826	...	...	·9824	·9822	·9829	·9824	·9855	·9856	·9857	·9844	·9827 „ ·9847
32000	·94360	·9919	·9916	...	...	·9915	·9917	·9916	·9915	·9929	·9931	·9934	·9926	·9918 „ ·9928
33912	1·00000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000 „ 1·0000

Ratios of Volumes of Liquid at Corresponding Pressures to Critical Volumes.

Pressures C_6H_5F.	Fraction of Critical Pressure.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$	CH_3OH.	C_2H_5OH.	C_3H_7OH	CH_3COOH	Ten Esters.
20	·00059	·3381	·3366	·3360	·3357	·3359	...	·3314	...	...	·3457	·3446	·3393	·3292 to ·3324
50	·00147	·3449	·3431	·3424	·3421	·3428	·3444	·3387	...	·3413	·3515	·3501	·3460	·3358 „ ·3392
100	·00295	·3508	·3488	·3482	·3480	·3489	·3505	·3447	...	·3455	·3565	·3552	·3523	·3419 „ ·3451
200	·00590	·3578	·3557	·3551	·3550	·3560	·3574	·3519	...	·3522	·3626	·3610	·3597	·3490 „ ·3523
400	·01180	·3661	·3641	·3635	·3635	·3648	·3660	·3611	·3631	·3596	·3698	·3679	·3686	·3576 „ ·3608
760	·02241	·3755	·3737	·3732	·3732	·3742	·3756	·3705	·3729	·3680	·3781	·3759	·3788	·3671 „ ·3704
1500	·04423	·3885	·3867	·3862	·3863	·3870	·3889	·3838	·3859	·3793	·3893	·3870	·3924	·3807 „ ·3836
3000	·08846	·4067	·4046	·4041	·4042	·4053	·4072	·4021	·4033	·3949	·4047	·4028	·4106	.3992 „ ·4021
5000	·14740	·4250	·4228	·4222	...	·4233	·4251	·4199	·4209	·4104	·4223	·4193	·4284	·4176 „ ·4205
7000	·20640	·4404	·4386	·4380	...	·4389	·4402	·4356	·4368	·4243	·4367	·4340	·4439	·4334 „ ·4364
10000	·29490	·4622	·4603	...	...	·4602	·4623	·4575	·4596	·4438	·4563	·4550	·4655	·4552 „ ·4583
15000	·44230	·4977	...	...	...	·4960	·4983	·4948	·4948	·4760	·4891	·4897	·5003	·4912 „ ·4940
20000	·58990	·5363	...	...	...	·5353	·5377	·5346	·5324	·5131	·5286	·5280	·5387	·5305 „ ·5354
25000	·73720	·5855	...	...	...	·5845	·5874	...	·5833	·5601	·5772	·5775	·5871	·5800 „ ·5855
28000	·82570	·6260	...	...	...	·6250	·6297	...	·6261	·6010	·6147	·6177	·6302	·6200 „ ·6254
30000	·88460	·6631	...	...	...	·6613	·6654	...	·6628	·6359	·6537	·6550	·6651	·6559 „ ·6633
32000	·94360	·7134	...	...	...	·7113	·7121	...	·7171	·6906	·7070	·7077	·7144	·7099 „ ·7204
33912	1·00000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000 „ 1·0000

Ratios of Volumes of Saturated Vapour at Corresponding Pressures to Critical Volumes.

Pressures C_6H_5F.	Fraction of Critical Pressure.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH	Ten Esters.
20	·00059	...	...	...	...	...	...	...	...	...	...	...	2290	... ...
50	·00147	...	...	...	...	...	...	...	...	...	...	...	1010	... ...
100	·00295	...	...	...	...	...	...	...	...	891	825	...	549	... ...
200	·00590	...	...	...	...	...	...	...	...	463	429	...	298	... ...
400	·01180	...	...	...	...	...	...	...	192	239	223	221	127	... ...
760	·02241	...	...	...	...	...	...	...	105	130	121	119	90·3	... ...
1500	·04423	55·5	55·3	54·8	55·1	54·4	52·8	54·6	55·2	67·3	62·9	61·6	48·5	56·9 to 59·4
3000	·08846	28·4	28·65	28·4	28·45	28·2	27·4	28·1	28·3	34·2	32·1	31·1	25·5	29·2 „ 30·3
5000	·14740	17·15	17·35	17·25	...	17·25	16·75	16·95	17·1	20·4	19·3	18·6	15·7	17·4 „ 18·45
7000	·20640	12·2	12·35	12·30	...	12·35	11·95	12·1	12·15	14·4	13·65	13·15	11·3	12·5 „ 13·1
10000	·29490	8·37	8·43	...	...	8·49	8·19	8·41	8·38	9·82	9·33	9·01	7·91	8·56 „ 8·92
15000	·44230	5·35	...	...	...	5·35	5·23	5·25	5·36	6·17	5·76	5·52	5·16	5·41 „ 5·57
20000	·58990	3·73	...	...	...	3·72	3·67	3·64	3·74	4·23	3·90	3·84	3·68	3·73 „ 3·85
25000	·73720	2·71	...	...	...	2·69	2·66	...	2·70	2·98	2·80	2·73	2·68	2·69 „ 2·78
28000	·82570	2·22	...	...	...	2·20	2·17	...	2·21	2·40	2·31	2·25	2·20	2·21 „ 2·28
30000	·88460	1·91	...	...	...	1·91	1·88	...	1·93	2·08	1·99	1·96	1·91	1·91 „ 1·95
32000	·94360	1·63	...	...	...	...	1·62	...	1·61	1·76	1·65	1·69	1·62	1·61 „ 1·64
33912	1·00000	1·00	1·00	1·00	1·00	1·00	1·00	...	1·00	1·00	1·00	1·00	1·00	1·00 „ 1·00

In the first group the differences are not great, but the ratios for benzene and carbon tetrachloride are below, and those for stannic chloride and ether at low pressures are somewhat above the average. The influence of molecular weight and constitution on the ratios for the esters is fully discussed in the paper by Mr. Thomas and myself (Trans. Chem. Soc. lxiii. p. 1252), and it will be sufficient here to mention that there is a marked rise as the molecular weight increases. This is not the case, however, with the ratios of the volumes of liquid or of saturated vapour to the critical volumes. All the ratios appear to depend to some extent on the constitution of the esters.

Ratios of Volumes of Liquid at Corresponding Pressures to Critical Volumes.—The differences in this case are small; in general the ratios for acetic acid are the highest, and those for the esters the lowest. At the lowest pressures the values for the alcohols are higher than for the esters, but at high pressures they are lower than for any of the other substances, and this is especially the case with methyl alcohol.

Ratios of Volumes of Saturated Vapour at Corresponding Pressures to Critical Volumes.—The grouping of the compounds is well seen in this case, the differences being naturally most marked at low pressures. Acetic acid stands quite alone with the lowest ratios, whilst the alcohols, and especially methyl alcohol, have much higher ratios than the other compounds. Of the remaining substances the esters have higher values than the members of the first group.

There is little doubt that the low ratios for acetic acid are due to the existence of complex molecules in the saturated vapour at low temperatures. The densities of the saturated vapours of the alcohols, on the other hand, are normal at low temperatures, and complex molecules cannot, therefore, be present; but there is considerable evidence of their existence in the liquid state.

Ramsay and Shields (Phil. Trans. 184 A. p. 647; Trans. Chem. Soc. lxiii. p. 1089) have recently studied the surface-energy of a large number of compounds, and have described a method by which the molecular complexity of liquids may be ascertained. They show that most of the liquids investigated have the same molecular weight in the liquid as in the gaseous state, but that there is greater molecular complexity in the liquid state in the case of the fatty acids and alcohols. As regards homologous compounds—both acids and alcohols—the complexity diminishes with rise of molecular weight, and as regards individual compounds it diminishes with rise of temperature; in order of complexity methyl alcohol comes next to acetic acid.

These conclusions, based on totally different considerations, agree perfectly with those suggested in this paper and in the previous one on the same subject.

An explanation of the relatively high molecular volumes of the saturated vapour of the alcohols at low temperatures would be afforded by the assumption that complex molecules exist to some extent at the critical points, an assumption which is supported by the high critical densities.

It has been shown (Phil. Mag. Nov. 1890, p. 423) that if the generalizations of Van der Waals were strictly true, the following relation should be true for all substances :—

$$\frac{v}{v'}=\frac{V}{V'}=\frac{T}{T'}\times\frac{p'}{p},$$

where v and v' are the molecular volumes of saturated vapour, V and V′ those of liquid, and T and T′ the boiling-points on the absolute scale of temperature of any two substances at corresponding pressures p and p'. This relation should hold good at the critical point, or

$$\frac{PV}{T}=\text{constant},$$

where P, V, and T are the critical pressure, the critical molecular volume, and the absolute critical temperature of any substance. As the critical volumes are now known, it is possible to test this relation at the critical points of the various compounds, and the values of $\frac{PV}{T}$ are given in the table below.

Again, the ratios of the actual to the theoretical density (for a perfect gas) at the critical point should be the same for all substances if Van der Waals's generalizations were strictly true: these ratios are also given in the table under the heading $\frac{D}{D'}$.

Substance.	$\frac{PV}{T}$.	$\frac{D}{D'}$.
Fluorbenzene	16390	3·79
Chlorobenzene	16380	3·77
Bromobenzene	16270	3·80
Iodobenzene	16360	3·78
Benzene	16610	3·71
Carbon tetrachloride	16940	3·65
Stannic chloride	16630	3·76
Ether	16290	3·83
Methyl alcohol	13750	4·52
Ethyl alcohol	15470	4·02
Propyl alcohol	15460	4·02

Substance.	$\frac{PV}{T}$.	$\frac{D}{D'}$.
Acetic acid	12440	5·00.
Methyl formate	15840	3·93
Ethyl formate	15990	3·88
Methyl acetate	15750	3·94
Propyl formate	16070	3·87
Ethyl acetate	15730	3·95
Methyl propionate	15920	3·90
Propyl acetate	15760	3·94
Ethyl propionate	15830	3·92
Methyl butyrate	15900	3·90
Methyl isobutyrate	16070	3·87

Here, again, the grouping of the compounds is well marked: acetic acid has by far the lowest value of $\frac{PV}{T}$ (and the highest density ratio); the alcohols come next in order, the values for methyl alcohol standing about midway between those for the other alcohols and acetic acid; the esters agree well together, the values of $\frac{PV}{T}$ being in every case lower than for the members of the first group.

Discussion.

Prof. Ramsay was not sure that the existence of complexes would alter the molecular volume in the liquid state; for liquids seem very compact. Experiments on the surface-energy of liquids had proved that complex molecules do exist in the alcohols and acetic acid. Dr. Young's conclusion was, therefore, confirmed by experiments of an entirely different nature.

Prof. Herschel was gratified to see Van der Waals's theory so well borne out in liquids, and hoped to see it extended to solids. The recent researches of Prof. Roberts-Austen on alloys seemed to point in this direction.

Mr. Rogers said molecular complexes do exert an influence on the properties of substances, as had been shown by Prof. Thorpe's viscosity experiments. Van der Waals's generalizations should therefore be looked at from a chemical as well as from a physical point of view.

The President thought the numbers brought forward showed fair agreement, especially when it was remembered that Van der Waals took no account of the complex molecules. Contrary to Prof. Ramsay, he would rather expect aggregation to affect the molecular volumes in the liquid state, for only about one-fifth the space was supposed to be occupied by matter. On the other hand, the relatively small contraction of liquids on cooling did not support this view.

XXXI. *The Action of Electromagnetic Radiations on Films containing Metallic Powders.* *By* Professor G. M. Minchin, *M.A.**

At the last Edinburgh meeting of the British Association attention was directed by Dr. Dawson Turner to a discovery made by M. E. Branly, as to the effect produced on a glass tube filled with copper filings through the extremities of which tube are inserted two wires which dip into the filings (without, of course, touching each other inside the tube). The filings form, apparently, a continuous column of metal, the tube when held up to the light being opaque. Supporting the tube in a horizontal, or any other, position, and connecting its terminal wires with a galvanometer and a voltaic cell in a continuous circuit, the chances are that no current whatever will be indicated by the galvanometer. If such a current happens to exist, a very slight tap given to the tube or to its support will destroy the conductivity of the column of filings, and no current passes. Another such tap will cause the column to conduct, and so on.

At first sight it seems strange that such a metallic column should be a non-conductor; but, of course, when we see the result we can admit that the contact between every particle and its neighbours is of the microphonic kind, and thus that the whole column may be an extremely bad conductor.

The identity of the state of affairs in the tube with that in a microphone may be perceived by inserting a telephone in the circuit, and then we shall find that slight mechanical disturbances communicated to the column of filings will be accompanied by the rumbling noises which are heard in a telephonic circuit which contains a microphone.

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* Read November 24, 1893.

Now the most striking peculiarity of such a column is that, while it is in the non-conducting state, if a spark is allowed to pass anywhere in its neighbourhood between the knobs of a Leyden jar or the poles of an induction-coil, the waves sent out from the spark at once render the column a conductor.

This experiment was reproduced at the meeting of the Physical Society on October 27 by Mr. Croft (*antea*, p. 413), and it at once struck me as bearing a strong resemblance to the effects produced by electromagnetic waves on photoelectric impulsion-cells. These cells I have already fully described (see Phil. Mag. March 1891); and in addition to what I showed with regard to them, I may here add that the effects of the sparks of an induction-coil on an impulsion-cell were observed when the cell (enclosed in a dark box) was over 140 feet distant from the coil and from everything connected with it. Moreover, it was found that, *in most cases*, the power of electromagnetic radiations to change the state of the cell from one of insensitiveness to one of sensitiveness to light depends on the length of the electromagnetic waves; for, by adding capacity (by means of Leyden jars) to the sparking circuit, the waves ceased to affect the cell, and, on removing this capacity, their ability to effect the change was restored.

I have always held that this action of the cell is due to a rearrangement of the molecules on the sensitized surface of the plate in the cell by the electrical disturbances produced in the wires connected with its poles, these disturbances being due to the radiations emanating from the sparking circuit; and I found that by completely enclosing the cell, *together with its terminal wires*, in a metal box, the ability of the radiations to act on the cell was destroyed.

After having seen the experiment shown by Mr. Croft, I filled several glass tubes with the filings of copper, tin, zinc, bismuth, antimony, &c., and found M. Branly's result in every case. It seems to me that if the filings are extremely fine (powders, in fact) the result is much more difficult to obtain. Powders used in this way are very great insulators, and they must be strongly pressed together by the corks which close the glass tubes in order to conduct; but then it becomes difficult to alter their conductivity by mechanical disturbances. On the other hand, a tube filled with very coarse iron filings was found to conduct under all circumstances; so that, apparently, the filings must be neither very fine nor very coarse.

As in the case of the impulsion-cells, the effect of the electromagnetic disturbances is cut off by completely enclosing the tubes and their terminal wires in a metal box. If the terminal wires are allowed to project outside the box—whether they are connected with the battery and galvanometer or not—the tubes are affected, which clearly shows that the result is due to the electrical disturbances set up in the leading wires of the tube.

In order to establish a closer connexion with the impulsion-cells, it seemed to me that films containing metallic particles almost in mathematical contact, but yet free to undergo very minute displacements of rotation, should replace the tubes of filings. Accordingly, I have formed such films by imbedding fine metallic powders in layers of gelatine and of collodion. The powders which I employ in these films are very much finer than the filings which I have used in the glass tubes.

The media, or bases, in which these powders are embedded are (so far as I have gone) gelatine and collodion.

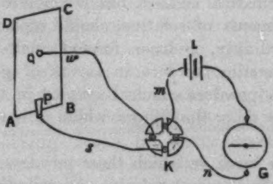
To form a gelatine film proceed thus:—Pour a very thin layer of dissolved gelatine over a glass or ebonite plate; let this layer get almost quite dry; if it gets quite dry, hold it over a beaker of boiling water until it absorbs a sufficient quantity of steam to render it very slightly plastic: take a test-tube with fine metallic powder immersed in alcohol; shake this up well, and rapidly pour some of it over the gelatine surface, securing, to all appearance, both continuity and uniformity of the metallic layer. The alcohol soon evaporates, leaving the metallic layer embedded in the gelatine, but not wholly sunk below it.

In the case of a collodion film, the metallic powder is shaken up in a test-tube and then rapidly poured over a glass plate. After some time the film dries and peels off the plate. The surface of the film which was in contact with the glass is the metallic surface sought; the other surface of the film probably contains no metallic particles, and is simply a continuous surface of dry collodion which is an absolute non-conductor.

If the layer of gelatine above described is too thick, it will probably happen that its upper surface (that remote from the glass or ebonite plate) does not contain metallic particles, and is a non-conductor. In this case, if the film is peeled off the plate there remains on the plate a metallic surface which will answer our purpose.

Such a film may now replace the tube of filings in M. Branly's experiment.

Let ABCD represent the film, L a battery of one or two voltaic cells, G a galvanometer, and K a key which can be connected with the film by means of the wires *s*, *w*, the former clamped to the film at a point P, while the latter ends in a stout platinum piece the end of which is a round knob touching the film at a movable point Q.



Now it is found that when the contacts are all made no current passes, and we have to begin by making the platinum knob touch the film very close to P—perhaps $\frac{1}{2}$ millim. from P. Still no current passes, even when a spark of an induction-coil is allowed to pass a few feet from the film. But by touching either of the wires *s*, *w* with an electrified body (that which I commonly use is a common gas-lighting electrical machine), the resistance of the film and the contacts

P, Q is overcome, and a strong current is shown by the galvanometer. We can now gradually increase the distance between P and Q each time that Q is removed from P, the wire *s* or *w* being touched by the electrified body; and by these successive steps the whole of the film is finally rendered conducting.

The special characteristic of such a film which I wish to point out is this: if, while the film is conducting, the current is stopped by breaking the contact at Q, the film will be found to be a non-conductor if the contact is almost instantly re-made at the same point Q; while, if the contact is broken anywhere else in the circuit, as at K, and then re-made, the film will still be found to conduct.

Again, if when the film has been rendered a conductor, the circuit is broken at K, and left broken for (apparently) any length of time, and then re-made, the film will be found to be still a conductor.

The breaking of the contact at Q is instantly fatal to the conductivity of the film, if the film is not many days old; but, after some days, it will be found that on breaking the contact at Q and then re-making it, the film remains a conductor, unless we delay the re-making of the contact for half a minute or so. Doubtless this is due to the fact that, as the film gets more hard and less plastic, the metallic particles find it more difficult to move, by rotation or otherwise, in their environment.

Finally, these films are very much less sensitive to mechanical disturbances given to their supports than the tubes of filings. They do not appear to be susceptible to the action of heat; but in many instances their conductivity was destroyed by breathing upon them, or by allowing a stream of steam to strike their surfaces, the conductivity being always restored by the electromagnetic radiations. The more rapid vibrations of light have not produced any effect so far as my observations have gone.

The prime cause of the action is to be sought in the electrical surgings produced in the leads *s* and *w*; but it is clear that the state of affairs at the places P, Q of contact of the film with the electrodes is a most important matter.

It seems clear, therefore, that the sensitized surface in an impulsion-cell has a close analogue in a slightly plastic film filled with almost mathematical completeness by fine metallic particles.

Discussion.

The President read a written communication from Prof. O. J. Lodge, in which the writer suggested that the phenomena of the films, and also of Lord Rayleigh's water-jet experiment (in which water drops are caused to coalesce by the presence of an electrified body), were due to the range of molecular attraction being increased by electric polarization.

Mr. Blakesley said he had tried Mr. Croft's experiments, and found that conductivity could be established in a tube of filings whilst the circuit was unclosed. Breaking the circuit of a transformer or electromagnet would produce conductivity; hence he concluded that electric pressure from a cell was not essential. Another curious experiment was to put the discharging knobs of an electric machine on a photographic plate at a distance of a few inches. On turning the machine a small spark travels slowly along the plate from the negative to the positive knob. On reversing the polarity of the machine, the spark travels back along the same path, but if the polarity remains unchanged, a second spark usually travels along a different path.

Prof. C. V. Boys asked Prof. Minchin whether the films themselves, or the contacts between the electrode and film, are made conducting by the sparks.

Prof. S. P. Thompson wished to know if ordinary photographic dry plates would serve the purpose.

Mr. Evershed enquired whether the metal used as electrode made any difference.

Prof. Minchin, in his reply, maintained that the phenomena were due to electric impulses. He had not tried photographic plates, and had always used platinum for his electrodes.

XXXII. *On the Sudden Acquisition of Conducting-Power by a Series of Discrete Metallic Particles.* *By* Prof. OLIVER J. LODGE[*].

THE recent experiments of Mr. Croft and Prof. Minchin remind me of an observation I frequently made when engaged with the syntonic arrangement of Leyden-jar circuits, or sympathetic electric resonance. I found, if the knobs of the receiver were very close together, a weak battery and bell being in circuit, that the occurrence of a scintilla at the receiver frequently caused the bell to ring for some time, and in general to show signs that the knobs were in a state of feebly adhesive contact. It was just as if their surface-layers or skins had been broken through, or opened out, in such a manner as to increase the molecular range of a few of the closest superficial molecules and thereby to cause cohesion to set in at a distance considerably greater than the ordinary distance.

A phenomenon which may be similarly caused is that discovered by Lord Rayleigh, with regard to the effect of electrified sealing-wax near a vertical water-jet: drops which would otherwise have rebounded being thereby caused to cohere. And, by the method of two impinging jets, the necessary difference of potential was shown to be only a volt or two, since a pair of impinging jets ceased to rebound if connected to the opposite terminals of a Grove cell.

Once more, the effect first observed by Mr. Guitard (1850), and rediscovered by myself and the late J. W. Clark, concerning the adhesion of dust-particles or mist-globules in electrified air, is but a more violent variety of the same sort of effect. And the action of electricity on a steam-jet discovered by Robert von Helmholtz and afterwards by Shelford Bidwell, and worked at by Richarz, Aitken, J. J. Thomson, and others, may not be very different.

Thinking of the Lord Rayleigh variety of experiment as in many respects the most definite, it is natural to explain it electrolytically as due to the polarization of the water-drops, or the formation in each drop of molecular chains each with a negatively charged oxygen atom at its termination on one hemisphere and a positively charged hydrogen atom at its opposite extremity.

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[*] Read November 24, 1893, as a contribution to the discussion on Prof. Minchin's paper.

Such drops then colliding about the region of their poles would be attracted not only by their ordinary cohesive forces, but by electrical force also, and thus the effective molecular range would be increased and cohesion might set in over an unusual distance.

In some such way I have allowed myself to fancy that the adhesion of my knobs might be explained, and I suggest that the conductivity of a chain of metallic filings under an electric polarising influence may be due to something of the same cause.

That a tap should break the minute points of contact is likely enough, though I see no reason why another tap should restore communication,—if so it does.

At the last moment I write this and send it off in the hope that it may arrive in time to be read at the Meeting as a contribution to the discussion on these interesting experiments.

Nov. 23, 1893.

XXXIII. *On the Magnetic Shielding of Concentric Spherical Shells.* *By* A. W. Rücker, *F.R.S.*[*]

Special interest has of late attached to the problem of shielding magnetic forces by means of iron screens, and recently the Astronomer Royal has described the arrangement (designed by Messrs. Johnson and Phillips) by which he has protected the absolute magnetic instruments at Greenwich from the disturbances due to a dynamo.

In that case external space was protected against magnetic forces produced inside the enclosure.

The converse arrangement is sometimes adopted, and, as in Lord Kelvin's Marine Galvanometer, instruments are enclosed in iron and thus protected against external magnetic influence.

[*] Read November 24, 1893.

In the particular case when the iron shields are spherical shells, the method of attacking the mathematical problem involved is well known. Maxwell gives the working in the case of an internal space protected by a single shell (Ed. 3, vol. ii. p. 59), and also discusses the precisely analogous problem of the flow of currents interrupted by shells of different conductivity from the rest.

I believe, however, that a number of facts, which are of some practical importance, are not generally known, and I propose therefore to deal with them in the following paper, illustrating them by numerical examples.

In the earlier part of the paper the general theory is developed, chiefly for the sake of defining the notation used, though this course involves some recapitulation from Maxwell. The discussion of the best conditions for shielding, with which I mainly deal, is, I believe, new.

The principal point to which attention is directed is the calculation of the advantage gained by lamination, *i. e.* by using shields separated by air-gaps, like those employed by Mr. Christie, instead of a continuous mass of iron.

That lamination would be useful might be foreseen by the aid of the hydrodynamical analogue to the magnetic field.

If we suppose sources and sinks of equal power to be placed within an enclosure, we may weaken the flow outside, either by making the enclosure or part of it very permeable, or by making the bounding surface very impermeable.

In like manner in the magnetic problem there are two ways of reducing the internal field, viz. by enclosing the central region by shells of greater, or by shells of less, permeability than the surrounding space. Although we cannot in practice use any material of less permeability than the air, we can introduce into an iron shell layers of material less permeable than itself. These may tend to reduce the external field by preventing as it were the force from penetrating to the external shell.

This analogy has long suggested that laminated shells, in which both the permeability of the iron and the relative impermeability of other materials are utilized, may be the most effective shields; and we are thus led to enquire which is the best arrangement that can be made under specified conditions as to the magnitude of the space at our disposal or as to the weight of the shielding material used. The answer to this question is supplied by the following discussion.

As the formulæ are somewhat heavy, I shall for the most part confine myself to shells formed of the same material and separated by air-gaps.

The second problem, to the result of which I think it is desirable to draw attention, is that which defines the relations between the shielding exerted by a number of concentric shells, (1) on external space when the magnetic forces are produced within the shells; and (2) on the enclosed space when the magnetic forces are produced outside. A full discussion of this is given below.

The whole of the investigation is subject to three limitations. I have confined myself:—

(1) to concentric shells;

(2) to cases in which the equipotential surfaces are surfaces of revolution about a line through the common centre of the shells;

(3) to the case in which the permeability of each shell is constant.

As regards the first two limitations, I think it will be seen that the conclusions arrived at are capable of generalization in such a way as to enable a better approach to be made to good shielding arrangements than would be the case if the results of the discussion in the simple case of spherical shells were unknown. The third limitation no doubt affects the applicability of the formulæ to practice. In spite of this, however, I venture to think that they afford some useful guidance, and that at all events they help to put the practical problem in definite terms.

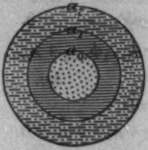
The Relation between the Shielded and Unshielded Fields when the Shielded Space is (1) *within,* (2) *without the Shielding Shells.*

Take the common centre of the shells as origin. Let the potential within any shell be expanded in terms of zonal spherical harmonics, then the terms corresponding to P_n will be of the form

$$\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n.$$

Each such term can be discussed independently, and it is more convenient to make the subscript numbers refer to the shell than to the order of the harmonic.

If, therefore, there are m shells, let the radii of the surfaces of separation be a_0, a_1, a_2, &c. a_m. Then a_0 is the radius of the central space, a_1, a_2, &c. are the external radii of consecutive shells (see figure).



Let $\mu_0, \mu_1, \ldots \mu_m$ be the permeabilities of the enclosed space and of the shells, and for generality let M be the permeability of external space.

Let the coefficient of P_n in the expansion of the potential in the pth shell be

$$\phi_p r^n + \psi_p / r^{n+1}.$$

Then at the boundary between the pth and the $(p+1)$th shells (of which the radius is a_p) the following conditions must be fulfilled:—

$$\phi_p a_p^n + \psi_p / a_p^{n+1} = \phi_{p+1} a_p^n + \psi_{p+1} / a_p^{n+1},$$

and

$$\mu_p \{ n\phi_p a_p^{n-1} - (n+1)\psi_p / a_p^{n+2} \}$$
$$= \mu_{p+1} \{ n\phi_{p+1} a_p^{n-1} - (n+1)\psi_{p+1} / a_p^{n+2} \};$$

or if we write

$$\mathrm{N} = (n+1)/n, \qquad \alpha_p = a_p^{-(2n+1)},$$

we get $$\phi_p + \psi_p \alpha_p - \phi_{p+1} - \psi_{p+1} \alpha_p = 0,$$

and $$\mu_p \phi_p - \mathrm{N}\mu_p \alpha_p \psi_p - \mu_{p+1}\phi_{p+1} + \mathrm{N}\mu_{p+1}\alpha_p \psi_{p+1} = 0.$$

If we suppose, for the sake of symmetry, and without reference for the moment to the expression of physical facts, that the coefficients of P_n in the potentials in the central enclosed space and in external space are

$$\phi_0 r^n + \psi_0 / r^{n+1} \quad \text{and} \quad \Phi r^n + \Psi / r^{n+1}$$

respectively, we have in all $2m+4$ quantities to deal with, and since the m shells have $m+1$ bounding surfaces, there are $2m+2$ equations between them, all of which are linear and of the types given above.

We thus get a series of equations as follows :—

$$\left.\begin{array}{l}
\Phi+\Psi a_m-\phi_m-\psi_m a_m=0. \\
\mathrm{M}\Phi-\mathrm{MN}\Psi a_m-\mu_m\phi_m+\mu_m\mathrm{N}\psi_m a_m=0. \\
\phi_m+\psi_m a_{m-1}-\phi_{m-1}-\psi_{m-1}a_{m-1}=0. \\
\mu_m\phi_m-\mu_m\mathrm{N}\psi_m a_{m-1}-\mu_{m-1}\phi_{m-1}+\mu_{m-1}\mathrm{N}\psi_{m-1}a_{m-1}=0. \\
\qquad \&c., \qquad\qquad\qquad\qquad \&c. \\
\phi_2+\psi_2 a_1-\phi_1-\psi_1 a_1=0. \\
\mu_2\phi_2-\mu_2\mathrm{N}\psi_2 a_1-\mu_1\phi_1+\mu_1\mathrm{N}\psi_1 a_1=0. \\
\phi_1+\psi_1 a_0-\phi_0-\psi_0 a_0=0. \\
\mu_1\phi_1-\mu_1\mathrm{N}\psi_1 a_0-\mu_0\phi_0+\mu_0\mathrm{N}\psi_0 a_0=0.
\end{array}\right\}(1)$$

In the case of shielding an enclosed space the potential of the inducing forces can be expanded within and in the neighbourhood of the shells in terms of positive powers of r, and since the potential of the induced magnetization must be finite when $r=0$, this also is included in ϕ_0. Hence $\psi_0=0$. In external space the expansion of the potential contains both positive and negative terms, the former due to the inducing, and the latter to the induced magnetization.

As the inducing forces are given Φ is known.

As $\psi_0=0$ and Φ is given, the number of unknowns is reduced by 2 and is equal to the number of the equations.

If, on the other hand, the inducing forces are produced in the central space within the shell, their potential must be expanded in inverse powers of r if the series is to be convergent at the boundary of the enclosure and beyond it. Hence ψ_0 is known. In external space the terms due both to the induced and inducing magnetizations are expressed in negative powers, and are included in Ψ. Thus $\Phi=0$.

Now ϕ_0/Φ is the ratio of the shielded to the unshielded field in the first case, in so far as it depends on the potential term under consideration, and Ψ/ψ_0 is the corresponding quantity when the forces are produced within the shell.

The relation between these may now be found.

Putting $\psi_0=0$ and writing Δ for the determinant formed by the coefficients of the ϕ's and ψ's in the above equations, except those which occur in the first and last columns, we have :—

$$\Phi\begin{vmatrix} 1, & \alpha_m, & -1, & -\alpha_m, & \dots\dots\dots & 0 \\ M, & -MN\alpha_m, & -\mu_m, & \mu_m N\alpha_m, & \dots\dots\dots & 0 \\ 0, & 0, & 1, & \alpha_{m-1}, & \dots\dots\dots & 0 \\ 0, & 0, & \mu_m, & -\mu_m N\alpha_{m-1}, & \dots\dots & 0 \\ & \dots\dots & \dots\dots & \dots\dots & & \\ 0, & 0, & 0, & 0, & \dots\dots\dots & -\alpha_1 \\ 0, & 0, & 0, & 0, & \dots\dots\dots & \mu_1 N\alpha_1 \\ 0, & 0, & 0, & 0, & \dots\dots\dots & \alpha_0 \\ 0, & 0, & 0, & 0, & \dots\dots & -\mu_1 N\alpha_0 \end{vmatrix} = \phi_0\Delta.$$

This may be written

$$\frac{\Phi\alpha_m}{M}\begin{vmatrix} M, & M, & -M, & \dots\dots\dots & 0 \\ M, & -MN, & -\mu_m, & \dots\dots\dots & 0 \\ 0, & 0, & 1, & \dots\dots\dots & 0 \\ & \dots\dots & \dots\dots & \dots\dots & \\ 0, & 0, & 0, & \dots & -\mu_1 N\alpha_0 \end{vmatrix} = \phi_0\Delta\ ;$$

whence, by subtracting the first from the second row,

$$\frac{\Phi\alpha_m}{M}\begin{vmatrix} M, & M, & -M & \dots\dots\dots & 0 \\ 0, & -M(N+1), & M-\mu_m, & \dots\dots & 0 \\ 0, & 0, & 1, & \dots\dots\dots & 0 \\ & \dots\dots & \dots\dots & \dots\dots & \\ 0, & 0, & 0, & \dots & -\mu_1 N\alpha_0 \end{vmatrix} = \phi_0\Delta.$$

$$\therefore\ -\Phi\alpha_m(N+1)M\begin{vmatrix} 1, & \alpha_{m-1}, & \dots\dots\dots & 0 \\ \mu_m, & -\mu_m N\alpha_{m-1}, & \dots\dots & 0 \\ & \dots\dots & \dots\dots & \\ 0, & 0, & 0, \dots & -\mu_1 N\alpha_0 \end{vmatrix} = \phi_0\Delta.$$

Proceeding in the same way with the new determinant, we finally get :—

$$\{-(N+1)^{m+1}\Phi\alpha_m\alpha_{m-1}\dots\alpha_0 M\mu_m\mu_{m-1}\dots\mu_1 = \phi_0\Delta.$$

Next, putting $\Phi=0$, we get in like manner

$$\Psi\Delta = \{-(\mathrm{N}+1)\}^{m+1}\psi_0\alpha_m\alpha_{m-1}\dots\alpha_0\mu_m\mu_{m-1}\dots\mu_1\mu_0\psi_0\,;$$

$$\therefore\quad \frac{\phi_0}{\Phi\mathrm{M}} = \frac{\Psi}{\mu_0\psi_0}.$$

Hence if $\mathrm{M}=\mu_0$, *i.e.* if the permeabilities of the enclosed and the external space are the same, the ratios of the shielded to the unshielded fields are the same *for each harmonic term* whether the shielded field be external or internal.

The most interesting application of this proposition is to the first harmonic term.

In this case the ϕ_0 and Φ are coefficients of terms of the form $r\mathrm{P}_1$, that is the field is uniform; and ψ_0 and Ψ are coefficients of terms of the form P_1/r^2, that is, the field is produced by a small magnet placed at the centre of the shells.

Hence the shielding effect on external space when a small magnet is placed at the centre of the shells is the same as the shielding effect on the enclosed space when the shells are placed in a uniform field.

In what follows I shall for the most part suppose that the magnetic forces are produced within the shells; but the above result enables the conclusions arrived at to be applied to the case of shielding against external forces.

(1) *Case of a Single Shell when the Permeabilities of the Internal and External Space are Unity.*

This case is well known, but may be included for the sake of completeness.

Since $\mu_0=\mathrm{M}=1$, the equations (1) reduce to

$$\begin{aligned}
\phi_1+\psi_1\alpha_1-\Psi\alpha_1 &= 0,\\
\mu_1\phi_1-\mathrm{N}\mu_1\alpha_1\psi_1+\mathrm{N}\alpha_1\Psi &= 0,\\
\phi_0+\psi_0\alpha_0-\phi_1-\psi_1\alpha_0 &= 0,\\
\phi_0-\mathrm{N}\alpha_0\psi_0-\mu_1\phi_1+\mathrm{N}\mu_1\alpha_0\psi_1 &= 0.
\end{aligned}$$

Hence Ψ/ψ_0 is the ratio of the shielded to the unshielded potential or, since this ratio is everywhere constant, that of the shielded to the unshielded field, in so far as it depends on the term under consideration, is

$$\frac{\alpha_0\mu_1(\mathrm{N}+1)^2}{(\mathrm{N}\mu_1+1)(\mathrm{N}+\mu_1)\alpha_0-\mathrm{N}\alpha_1(\mu_1-1)^2}. \quad . \quad . \quad . \quad (2)$$

If the unshielded field is due to a small magnet placed at the centre of the sphere, the only term in the expression for the potential is

$$\mathrm{A}_1\mathrm{P}_1/r^2.$$

Hence

$$n=1, \quad \mathrm{N}=(n+1)/n=2, \quad \alpha_0=a_0^{-(2n+1)}=a_0^{-3}, \quad \&\mathrm{c}.$$

Then the above ratio becomes

$$\frac{9a_1^3\mu_1}{(2\mu_1+1)(2+\mu_1)a_1^3-2(\mu_1-1)^2a_0^3}=\frac{9a_1^3\mu_1}{9a_1^3\mu_1+2(\mu_1-1)^2(a_1^3-a_0^3)}.$$

If μ_1 is so great that a small integer may be neglected with regard to it, this reduces to

$$\frac{9}{2\mu_1}\,\frac{a_1^3}{a_1^3-a_0^3};$$

but this approximate form is only valid if a_1 and a_0 are not nearly equal.

It will be observed that the shielding depends only on the ratio of $a_1 : a_0$, and not on the absolute dimensions.

If the shell is thin, let $a_0=a_1-t$. Then, if squares and higher powers of t are neglected,

$$\frac{\Psi}{\psi_0}=\frac{3\mu_1}{3\mu_1+2(\mu_1-1)^2\dfrac{t}{a_1}}.$$

If μ_1 is not large this may be written

$$\frac{\Psi}{\psi_0}=1-\frac{2(\mu_1-1)^2}{3\mu_1}\cdot\frac{t}{a_1}.$$

If μ_1 is large it becomes

$$\frac{\Psi}{\psi_0}=\frac{1}{1+\frac{2}{3}\mu_1\dfrac{t}{a_1}}.$$

In this approximation we have neglected terms of the order $\mu_1 t^2/a_1^2$. The expression

$$\frac{\Psi}{\psi_0} = 1 - \frac{2}{3}\mu_1 \frac{t}{a_1}$$

is only valid if terms of the order $\mu_1^2 t^2/a_1^2$ are negligible; and as μ_1 is by supposition large, they may be very much greater than those which have already been rejected.

As an example, if $t/a_1 = 0{\cdot}01$, the ratio of the shielded to the unshielded field is 3/13 if $\mu_1 = 500$, and 3/23 if $\mu_1 = 1000$.

The following Table of Values is calculated for cases in which $a_1 - a_0$ is not small, on the assumption that $\mu_1 = 1000$. The corresponding numbers can be obtained in any other case when μ_1 is large by remembering that the ratio under investigation varies inversely as μ_1.

a_0/a_1.	Ψ/ψ_0.
0·9	1/60
0·8	1/108
0·7	1/146
0·6	1/174
0·5	1/194

This table illustrates the difficulty of improving the shielding by increasing the thickness of a shield already moderately thick. When the latter is increased in the ratio 5 : 1, and the internal empty space is reduced sixfold, the shielded field is only reduced in the ratio of about 1 : 3.

In general, if the volume of the material employed in a shell is seven times that of the internal spherical space the external field will be reduced to 1/100, 1/200, or 1/300 of its unshielded value, according as the permeability of the material is 500, 1000, or 1500.

Next consider the case of a small magnet placed at a distance b from the centre of the shell.

If the axis of the magnet lies on a diameter, and if the moment is ψ_0, the unshielded potential is

$$\psi_0 \left\{ \frac{P_1}{r^2} + \frac{2P_2 b}{r^3} + \frac{3P_3 b^2}{r^4} + \&c. + \frac{nP_n b^{n-1}}{r^{n+1}} + \&c. \right\},$$

where r is measured from the centre of the sphere.

To find the potential in the shielded field, we must multiply each term in the expression by the corresponding value of

$$\frac{\alpha_0\mu_1(\mathrm{N}+1)^2}{(\mathrm{N}\mu_1+1)(\mathrm{N}+\mu_1)\alpha_0-\mathrm{N}\alpha_1(\mu_1-1)^2},$$

which may be called the *shielding factor*.

If the shell is thin, so that $a_0=a_1-t$, where t is small, this factor (remembering that $\alpha_0=a_0^{-(2n+1)}$) becomes

$$\frac{a_1^{2n+1}\mu_1(\mathrm{N}+1)^2}{a_1^{2n+1}\mu_1(\mathrm{N}+1)^2+(2n+1)\mathrm{N}(\mu_1-1)^2a_1^{2n}t}$$

$$=\frac{(2n+1)\mu_1}{(2n+1)\mu_1+n(n+1)(\mu_1-1)^2t/a_1},$$

which, if μ_1 is large, simplifies to

$$\frac{2n+1}{2n+1+n(n+1)\mu_1t/a_1}.$$

Calling these shielding factors S_1, S_2, &c., the unshielded and shielded forces in the direction of the radius are given by

$$-\psi_0\left\{\frac{2\mathrm{P}_1}{r^3}+\frac{2\,.\,3\mathrm{P}_2b}{r^4}+\frac{3\,.\,4\,.\,\mathrm{P}_3b^2}{r_5}+\&\text{c.}\right\}$$

and by

$$-\psi_0\left\{\frac{2\mathrm{P}_1\mathrm{S}_1}{r^3}+\frac{2\,.\,3\,.\,\mathrm{P}_2b\mathrm{S}_2}{r^4}+\frac{3\,.\,4\,.\,\mathrm{P}_3b^2\mathrm{S}_3}{r^5}+\&\text{c.}\right\}$$

respectively.

If we consider any point on the diameter on which the axis of the magnet lies and on the side towards which it is moved, all the P's are equal to unity.

The larger b becomes, the more important are the terms in which the shielding factors S_2, S_3 are small, and the more efficient is the shielding. The total force is, however, increased, as all these terms are added to the first. In other words, the force is increased owing to the approach of the magnet to the point considered, and it is only the addition to its original value which is better shielded. The average shielding is improved, but the force is increased.

The converse statement holds good for the side from which the magnet is moved. In that case the P's are alternately

-1 and $+1$, the shielding is less efficient, but the force is diminished.

It may be worth while to illustrate this by a numerical example. Let the magnet be placed on a diameter at a distance from the centre = half the external radius; and let the forces be calculated at points on the same diameter, the distances of which from the exterior of the shell are also half the external radius.

Then

$$b=a_1/2, \quad r=3a_1/2.$$

Let the thickness of the shield be 0·01 of the radius, so that $t/a_1=0{\cdot}01$, and let the permeability be 500.

The unshielded force is :—

$$-\frac{\psi_0}{a_1^3}\{0{\cdot}593\,P_1+0{\cdot}593\,P_2+0{\cdot}395\,P_3+0{\cdot}219\,P_4+0{\cdot}100\,P_5 \\ +0{\cdot}051\,P_6+0{\cdot}023\,P_7+0{\cdot}010\,P_8+\&c.\}.$$

Multiplying each term by the corresponding shielding factor, the shielded force is :—

$$-\frac{\psi_0}{a_1^3}\{0{\cdot}137\,P_1+0{\cdot}085\,P_2+0{\cdot}041\,P_3+0{\cdot}018\,P_4+0{\cdot}008\,P_5 \\ +0{\cdot}003\,P_6+0{\cdot}0011\,P_7+0{\cdot}0004\,P_8\}.$$

The true values of the unshielded forces are

$$-2\psi_0/a_1^3 \quad \text{and} \quad 2\psi_0/(2a_1)^3=0{\cdot}25\psi_0/a_1^3.$$

The above series gives

$$-1{\cdot}994\psi_0/a_1^3 \quad \text{and} \quad 0{\cdot}248\psi_0/a_1^3,$$

which proves that the number of terms taken gives results sufficiently approximate for the purpose of illustration.

The corresponding values of the shielded forces are

$$-0{\cdot}294\psi_0/a_1^3 \quad \text{and} \quad 0{\cdot}081\psi_0/a_1^3.$$

These conclusions may be summed up as follows :—

Let A and B be the two external points, and let A be that towards which the magnet is moved. Then we get the results given in the following table :—

Position of Magnet.	Numbers proportional to force when		Ratio.
	Unshielded.	Shielded.	
Centre	0·593 at A & B	0·137	0·23
Halfway between centre and outer boundary of shell.	2 at A	0·294	0·15
	0·25 at B	0·081	0·33

Case of Two or Three Concentric Shells of the same Permeability separated by Air-Gaps.

In the case of three shells there are twelve simultaneous equations, and we finally get :—

$$\left.\begin{aligned}\Psi\,[&\xi\eta(\eta\alpha_1-\xi\alpha_0)\alpha_1\alpha_4(\alpha_4-\alpha_5)(\alpha_2-\alpha_3)\\&+\xi\eta(\eta\alpha_2-\xi\alpha_3)\alpha_2\alpha_4(\alpha_0-\alpha_1)(\alpha_4-\alpha_5)\\&+\xi\eta(\eta\alpha_5-\xi\alpha_4)\alpha_2\alpha_3(\alpha_0-\alpha_1)(\alpha_2-\alpha_3)\\&+\alpha_1\alpha_3(\xi\alpha_3-\eta\alpha_3)(\xi\alpha_4-\eta\alpha_5)(\xi\alpha_0-\eta\alpha_1)]\\&=\psi_0\alpha_0\alpha_1\alpha_2\alpha_3\alpha_4(\mathrm{N}+1)^6\mu^3,\end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

where $\xi=(\mathrm{N}\mu+1)(\mathrm{N}+\mu)$ and $\eta=\mathrm{N}(\mu-1)^2$.

This reduces to the case of two shells if we put any two consecutive α's equal to each other ; and it will be found that in all these cases the expression assumes the same form.

If $\alpha_4=\alpha_5$, remembering that $\xi-\eta=\mu(\mathrm{N}+1)^2$,

$$\left.\begin{aligned}\Psi\{-\xi\eta\alpha_2(\alpha_0-\alpha_1)(\alpha_2-\alpha_3)+\alpha_1(\xi\alpha_2-\eta\alpha_3)(\xi\alpha_0-\eta\alpha_1)\}\\=\psi_0\alpha_0\alpha_1\alpha_2\mu^2(\mathrm{N}+1)^4\\=\Psi\mathrm{D}\text{ say.}\end{aligned}\right\} \quad . \quad (4)$$

Then

$$\left.\begin{aligned}\frac{\mathrm{D}^2d\Psi}{\psi_0\mu^2(\mathrm{N}+1)^4}=&\{\xi\eta\alpha_2(\alpha_2-\alpha_3)-\eta\alpha_1(\xi\alpha_2-\eta\alpha_3)\}\alpha_1^2\alpha_2d\alpha_0\\&+\{-\xi\eta\alpha_2\alpha_0(\alpha_2-\alpha_3)+\eta\alpha_1^2(\xi\alpha_2-\eta\alpha_3)\alpha_0\alpha_2d\alpha_1\\&+\{\xi\eta\alpha_2^2(\alpha_0-\alpha_1)-\eta\alpha_1\alpha_3(\xi\alpha_0-\eta\alpha_1)\}\alpha_0\alpha_1d\alpha_2\\&+\{-\xi\eta\alpha_2(\alpha_0-\alpha_1)+\eta\alpha_1(\xi\alpha_0-\eta\alpha_1)\}\alpha_0\alpha_1\alpha_2d\alpha_3.\end{aligned}\right\} \quad (5)$$

As α_0 and α_3 do not occur in the coefficients of $d\alpha_0$ and $d\alpha_3$ respectively, it is evident that no particular values of either these quantities give a maximum or minimum value of Ψ when the other three α's are arbitrarily selected.

Again, the coefficient of $d\alpha_0$ may be written

$$\eta\alpha_1^2\alpha_2\{\alpha_2(\xi\alpha_2-\xi\alpha_3)-\eta\alpha_1(\xi\alpha_2-\eta\alpha_3)\}.$$

Now $\alpha_2<\alpha_1$, and since ξ is $>\eta$,

$$\xi\alpha_2-\xi\alpha_3 \text{ is } <\xi\alpha_2-\eta\alpha_3.$$

Hence this expression is always negative, *i. e.* Ψ diminishes as α_0 increases, that is as the radius a_0 diminishes.

Thus adding the permeable material within the shell so as to reduce the internal cavity always improves the shielding.

In like manner, by discussing the coefficient of $d\alpha_3$, it may be shown that an improvement is always effected by increasing the external diameter.

If, however, the smallest and largest radii are given, Ψ may be a maximum or minimum if the coefficients of $d\alpha_1$ and $d\alpha_2$ vanish ; *i. e.* if

$$\alpha_1^2=\frac{\xi\alpha_2\alpha_0(\alpha_2-\alpha_3)}{\xi\alpha_2-\eta\alpha_3},$$

and

$$\alpha_2^2=\frac{\alpha_1\alpha_3(\xi\alpha_0-\eta\alpha_1)}{\xi(\alpha_0-\alpha_1)}.$$

Let us first suppose that α_1 is also given, *i. e.* that the dimensions of the inner shell are fixed, then the equations enable us to determine the effect of adding another external shell of which the outer radius is also given.

We may first suppose the external shell to be of vanishing thickness, in which case $\alpha_2=\alpha_3$, and then allow α_2 to increase, *i. e.* the inner radius of the outer shell (a_2) to diminish.

If $\alpha_2=\alpha_3$ the coefficient of $d\alpha_2$ is negative, for $\alpha_3<\alpha_1$ and $\xi(\alpha_0-\alpha_1)<\xi\alpha_0-\eta\alpha_1$.

Hence the addition of a thin external shell reduces Ψ, *i. e.* improves the shielding.

This improvement will reach a maximum when

$$\alpha_2^2=\frac{\alpha_1\alpha_3(\xi\alpha_0-\eta\alpha_1)}{\xi(\alpha_0-\alpha_1)};$$

but in order that this may correspond to a physical reality we must have

$$\alpha_1 > \alpha_2 > \alpha_3.$$

The first of these inequalities when combined with the above equation is equivalent to

$$\alpha_1\xi(\alpha_0-\alpha_1) > \alpha_3(\xi\alpha_0-\eta\alpha_1).$$

As α_1 is $>\alpha_3$ and $\xi(\alpha_0-\alpha_1) < \xi\alpha_0-\eta\alpha_1$, this condition may be, but is not necessarily, fulfilled.

The second is equivalent to

$$\alpha_1(\xi\alpha_0-\eta\alpha_1) > \alpha_3\xi(\alpha_0-\alpha_1),$$

which is always true.

Hence we conclude that if the external radius of the outer shell is so chosen that

$$\alpha_3 < \alpha_1\frac{\xi(\alpha_0-\alpha_1)}{\xi\alpha_0-\eta\alpha_1},$$

there will be a maximum value of the shielding, while an air-space intervenes between the two shields. There will not be a maximum if α_3 is greater than this limit. In the case of the first harmonic the condition for a maximum becomes

$$a_3 > a_1\left\{\frac{\xi a_1^3-\eta a_0^3}{\xi(a_1^3-a_0^3)}\right\}^{\frac{1}{3}};$$

where a_0, a_1, and a_3 are the radii of the corresponding surfaces. If there is such a maximum it is evident that for some thickness of the external shell less than that which gives the best result the shielding must be the same as if the whole of the air-gap were filled.

To find this value of α_2 we have to put $\alpha_2=\alpha_1$ in the expression for Ψ, and equate the result to the value of Ψ when α_2 is not$=\alpha_1$.

Remembering that $\mu(\mathrm{N}+1)^2=\xi-\eta$ we get

$$\frac{\alpha_0}{(\xi\alpha_0-\eta\alpha_3)(\xi-\eta)} = \frac{\alpha_0\alpha_1\alpha_2}{-\xi\eta\alpha_2(\alpha_0-\alpha_1)(\alpha_1-\alpha_3)+\alpha_1(\xi\alpha_2-\eta\alpha_3)(\xi\alpha_0-\eta\alpha_1)}.$$

This leads to a quadratic in α_2, of which one root must evidently be $\alpha_2=\alpha_1$.

The other root is

$$a_2 = a_3 \frac{\xi a_0 - \eta a_1}{\xi(a_0 - a_1)}.$$

If then we write $I = (\xi a_0 - \eta a_1)/\xi(a_0 - a_1)$, where I is a function only of the dimensions of the inner shell, of the permeability of the material and of the order of the harmonic term considered, we have reached the following conclusions.

If $a_3 I > a_1$, the shielding improves continuously as the thickness of the external shell increases from without inwards.

If $a_3 I < a_1$, the shielding at first improves and then deteriorates as the thickness increases.

When $a_2 = a_3 I$, the shielding is the same as when the whole of the space between the shells is filled, *i. e.* as when $a_2 = a_1$.

When $a_2^2 = a_1 a_3 I$, the shielding is a maximum, after which it diminishes until $a_2 = a_1$.

In the particular case when the maximum shielding is equal to that when the hollow space is just filled, the conditions $a_2 = a_3 I$ and $a_2^2 = a_1 a_3 I$ must be simultaneously fulfilled. Hence $a_2 = a_1$, and there is only one maximum, which occurs when the hollow space is just filled.

This can only be the case when

$$a_2^2 = a_1^2 = \frac{a_1 a_3 (\xi a_0 - \eta a_1)}{\xi(a_0 - a_1)},$$

i. e., when

$$\xi a_1^2 - a_1(\xi a_0 + \eta a_3) + \xi a_0 a_3 = 0.$$

It may be worth while to give a numerical example to illustrate these results.

Let the small magnet be placed in the centre. Then the only harmonic term is that for which $n=1$ and $N=2$.

Let
$$a_0 = 8a_3 = 2a,$$
i. e.
$$a_3 = 2a_0 \text{ and } a_1 = 1{\cdot}260 a_0.$$
Let
$$\mu = 501.$$
Then
$$\xi = (N\mu + 1)(N + \mu) = 1003 \times 503,$$
$$\eta = N(\mu - 1)^2 = 2 \times 500^2 = \xi \text{ nearly};$$
or to a closer approximation,
$$\eta = \xi(1 - 0{\cdot}009).$$

Hence

$$I=\frac{\xi\alpha_0-\eta\alpha_1}{\xi(\alpha_0-\alpha_1)}=1+0{\cdot}009.$$

By substituting in (4) we get the values of Ψ/ψ_0 given in the following table for different values of α_2. The volumes are expressed in terms of the space enclosed by the innermost surface.

$a_2=$	$a_2=$	Volume of outer shell.	Total Volume.	Ψ/ψ_0.
$a_3=a_0/8$.	$a_3=2a_0$.	0	1·00	0·018
$1{\cdot}009a_3=0{\cdot}1261a_0$.	$0{\cdot}997a_3=1{\cdot}994a_0$.	0·07	1·07	0·0102
$3a_3/2=3a_0/16$.	$0{\cdot}873a_3=1{\cdot}746a_0$.	2·66	3·66	0·0007
$2a_3\doteq a_0/4$.	$0{\cdot}794a_3=1{\cdot}588a_0$.	4·00	5·00	0·0006
$3a_3=3a_0/8$.	$0{\cdot}693a_3=1{\cdot}386a_0$.	5·33	6·33	0·0009
$a_1=4a_3=a_0/2$.	$0{\cdot}630a_3=1{\cdot}260a_0$.	6·00	7·00	0·0102

Turning next to the case when α_1 and α_2 both vary, we may determine the maximum shielding for given external and internal radii.

The conditions to be fulfilled are:

$$\left.\begin{aligned}-\xi\alpha_2\alpha_0(\alpha_2-\alpha_3)+\alpha_1^2(\xi\alpha_2-\eta\alpha_3)=0\\ \xi\alpha_2^2(\alpha_0-\alpha_1)-\alpha_1\alpha_3(\xi\alpha_0-\eta\alpha_1)=0\end{aligned}\right\} \quad . \quad . \quad (6)$$

Adding these we get either

$$\alpha_2=\alpha_1 \quad \text{or} \quad \alpha_2\alpha_1=\alpha_0\alpha_3.$$

The first has reference only to the point at which the maximum value is equal to that produced when all the space between the shells is filled.

The latter leads to the equation

$$\xi\alpha_2^3(\alpha_2-\alpha_3)-\alpha_3^2\alpha_0(\xi\alpha_2-\eta\alpha_3)=0.$$

This equation is a biquadratic in α_2.

Since the expression is positive when $\alpha_2=-\infty$ and negative when $\alpha_2=\alpha_3$, there is at least one real root $<\alpha_3$, with which, since $\alpha_2>\alpha_3$, we are not concerned.

As regards roots $>\alpha_3$, it is evident, since the whole expression is positive when $\alpha_2=\infty$, that there is at least one real positive root $>\alpha_3$. Also, as $\xi\alpha_2-\eta\alpha_3$ is positive, that there can only be one positive root $>\alpha_3$.

This root must be $<\alpha_1$, *i. e.* since $\alpha_2\alpha_1=\alpha_0\alpha_3$, $\alpha_2^2<\alpha_0\alpha_3$; *i. e.* the expression must be positive when $\alpha_2^2=\alpha_0\alpha_3$.

This is equivalent to the condition

$$\xi\sqrt{\alpha_0}(\sqrt{\alpha_0}-\sqrt{\alpha_3})>\sqrt{\alpha_3}(\xi\sqrt{\alpha_0}-\eta\sqrt{\alpha_3}).$$

As $\quad \alpha_0>\alpha_3$ and $\xi\sqrt{\alpha_0}-\xi\sqrt{\alpha_3}<\xi\sqrt{\alpha_0}-\eta\sqrt{\alpha_3}$,

this equality may be, but is not necessarily, satisfied.

Before discussing these equations further, it is convenient to modify their form.

Let

$$\frac{\alpha_0}{\alpha_1}=\frac{\alpha_2}{\alpha_3}=\lambda\,; \qquad \eta=\xi(1-\epsilon)\,; \qquad \frac{\alpha_0}{\alpha_3}=\mathrm{L}.$$

Then from (6),

$$\lambda^3(\lambda-1)-\mathrm{L}(\lambda-1+\epsilon)=0, \quad . \quad . \quad . \quad . \quad (7)$$

and from (4),

$$\Psi/\psi_0=\frac{\mu^2(\mathrm{N}+1)^4\lambda^2\mathrm{L}}{\xi^2\{(\lambda-1+\epsilon)^2\mathrm{L}-(1-\epsilon)\lambda^2(\lambda-1)^2\}}\,; \qquad (7a)$$

or from (7),

$$\Psi/\psi_0=\frac{\mu^2(\mathrm{N}+1)^4\mathrm{L}^2}{\xi^2(\lambda-1)^2\{\lambda^4-\mathrm{L}(1-\epsilon)\}}=\frac{\mathrm{L}^2\epsilon^2}{(\lambda-1)^2\{\lambda^4-\mathrm{L}(1-\epsilon)\}}, \quad (8)$$

since $\qquad \mu(\mathrm{N}+1)^2=\xi-\eta=\epsilon\xi.$

In equation (8) Ψ becomes infinite when $\lambda=1$ and when $\lambda^4=\mathrm{L}(1-\epsilon)$.

The first of these assumptions is inconsistent with equation (7), and by substituting from $\lambda^4=\mathrm{L}(1-\epsilon)$ in (7) we get

$$(\lambda-1+\epsilon)^2=-\epsilon(1-\epsilon),$$

which is only possible if either ϵ or $1-\epsilon$ is negative, which they can never be, as ξ is always $>\eta$, and η and ξ are always positive.

In the case when the permeability is very great, $\xi=\eta=\mathrm{N}\mu^2$ nearly and ϵ may be neglected in equation (7), so that $\lambda^3=\mathrm{L}$.

If, however, $\lambda - 1$ is very small, *i. e.* if the shells are very thin, the term in ϵ may be comparable with the others, and the approximation is not legitimate. In cases where it may be applied we get from equation (8) the very simple result

$$\Psi/\psi_0 = \frac{(N+1)^4}{N^2\mu^2}\frac{\lambda^3}{(\lambda-1)^3} = \frac{L\epsilon^2}{(\sqrt[3]{L}-1)^3}.$$

In this case also the four quantities α_0, α_1, α_2, α_3 are in Geometrical Progression.

If $\alpha_1 = \alpha_2$, $\alpha_0/\alpha_3 = \lambda^2 = L$, so that (7) reduces to

$$(\sqrt{L}-1)^2 = \epsilon, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

of which the root greater than unity is $\sqrt{L} = 1 + \sqrt{\epsilon}$.

This furnishes a limit below which the equations do not apply.

Below the limiting case lamination must be disadvantageous.

If we remove a thin slice either from the exterior or from the interior surface of a single shell, the shielding power is impaired; and if the shielding does not attain a maximum when this spherical crack is supposed to traverse the shell from inside to outside, it must always be less than if there is no crack.

Next, comparing the shielding effect of a single shell with that of a double shell of which the innermost and outermost radii are the same as that of the single shell, if we write in equation (2) α_3 for α_1, and put $\eta = \xi(1-\epsilon)$, we find by comparing with (4) that the single shell will shield best if

$$\frac{\alpha_0\epsilon}{\alpha_0-\alpha_3(1-\epsilon)} < \frac{\alpha_0\alpha_1\alpha_2\epsilon^2}{-(1-\epsilon)\alpha_2(\alpha_0-\alpha_1)(\alpha_2-\alpha_3)+\alpha_1\{\alpha_2-\alpha_3(1-\epsilon)\}\{\alpha_0-\alpha_1(1-\epsilon)\}},$$

which reduces to

$$\left(\frac{\alpha_2}{\alpha_3}-1\right)\left(\frac{\alpha_0}{\alpha_1}-1\right) < \epsilon.$$

Now if we diminish α_1, *i. e.* increase the outer radius of the inner shell, α_0/α_1 is increased; and in like manner by diminishing the inner radius of the outer shell α_2/α_3 is increased. Hence the largest value of the left-hand side of the inequality is attained when $\alpha_1 = \alpha_2$, so that if $\alpha_2/\alpha_3 = \lambda_1$, and $\alpha_0/\alpha_1 = \lambda$, $\lambda\lambda_1 = L$.

Thus

$$(\lambda_1-1)(\lambda-1)=\left(\frac{L}{\lambda}-1\right)(\lambda-1),$$

of which the largest value is $(\sqrt{L}-1)^2$.

If ϵ is greater than $(\sqrt{L}-1)^2$, lamination is injurious, the limit thus fixed being the same as that previously found.

Above this limit, if for a given α_0 and α_3 we choose α_1 and α_2 so that

$$\left(\frac{\alpha_2}{\alpha_3}-1\right)\left(\frac{\alpha_0}{\alpha_1}-1\right)<\epsilon,$$

the two shells will shield better than one, *i. e.* better than if the whole of the space between the innermost and outermost surfaces were filled with the permeable material.

It of course follows *à fortiori* that the best arrangement of the two shells will, under these circumstances, give better results than the single shell, a conclusion which can be verified directly.

For we must by (8) and (2) have

$$\frac{L^2\epsilon^2}{(\lambda-1)^2\{\lambda^4-L(1-\epsilon)\}}<\frac{L\epsilon}{L-(1-\epsilon)},$$

where

$$\lambda^3(\lambda-1)-L(\lambda-1+\epsilon)=0.$$

Eliminating L, the inequality becomes

$$(1-\epsilon)\{\epsilon-(\lambda-1)^2\}^2>0.$$

which, since ϵ is always positive and <1, must always be true.

At the limiting point the curves which give the relations between L and the Ψ's for one and two shells respectively touch each other.

For, as in the case of a single shell,

$$\frac{\Psi_0}{\psi}=\frac{\epsilon L}{L-(1-\epsilon)},$$

$$\frac{1}{\psi_0}\frac{d\Psi}{dL}=-\frac{\epsilon(1-\epsilon)}{\{L-(1-\epsilon)\}^2}.$$

At the limiting point $L=(1+\sqrt{\epsilon})^2$;

$$\therefore \frac{1}{\psi_0}\frac{d\Psi}{dL}=-\frac{1-\sqrt{\epsilon}}{4(1+\sqrt{\epsilon})}.$$

In the case of two shells we have from (7),

$$\{\lambda^2(4\lambda-3)-L\}\frac{d\lambda}{dL}=\lambda-1+\epsilon,$$

and since at the limiting point $\lambda^2=L=(1+\sqrt{\epsilon})^2$,

$$\frac{d\lambda}{dL}=\frac{1}{4(1+\sqrt{\epsilon})}.$$

Also from (8),

$$\frac{1}{\psi_0}\frac{d\Psi}{dL}=\frac{2L\epsilon^2}{(\lambda-1)^2\{\lambda^4-L(1-\epsilon)\}}+\frac{L^2\epsilon^2(1-\epsilon)}{(\lambda-1)^2\{\lambda^4-L(1-\epsilon)\}^2}$$

$$-2\left\{\frac{1}{\lambda-1}+\frac{2\lambda^3}{\lambda^4-L(1-\epsilon)}\right\}\frac{L^2\epsilon^2}{(\lambda-1)^2\{\lambda^4-L(1-\epsilon)\}}\frac{d\lambda}{dL};$$

and if we substitute in this the values of λ, L, and $d\lambda/dL$ proper to the limiting point, we get

$$\frac{1}{\psi_0}\frac{d\Psi}{dL}=-\frac{1-\sqrt{\epsilon}}{4(1+\sqrt{\epsilon})},$$

which is the same as the value obtained in the case of the single shell.

The shielding factor in the limiting case is obtained by putting $\lambda^2=L=(1+\sqrt{\epsilon})^2$ in (8), or in the corresponding expression for a single shell. In either case

$$\frac{\Psi}{\psi_0}=\frac{\sqrt{\epsilon}(1+\sqrt{\epsilon})}{2}.$$

Since

$$\epsilon=\frac{\xi-\eta}{\xi}=\frac{\mu(N+1)^2}{(N\mu+1)(N+\mu)},$$

which increases with N, *i. e.* with $(n+1)/n$, and therefore diminishes as n increases, the thickness of the shell at the limit is less for the higher harmonic terms, but the shielding at the limit is better.

If μ is so large, however, that we may write

$$\epsilon=\frac{(N+1)^2}{N\mu}=\frac{(2n+1)^2}{n(n+1)\mu},$$

the limiting thicknesses and the shielding factors for the principal harmonic terms vary very slowly. If $\mu=400$ we get :—

$n.$	$e.$	$\mathrm{L}=(1+\sqrt{e})^2.$	$a_1/a_0=(1+\sqrt{e})^{\frac{2}{3}}.$	$\Psi/\psi_0.$
1.	·0112	1·223	1·069	·058
2.	·0104	1·214	1·067	·056
3.	·0102	1·212	1·066	·055

Thus the limiting thickness in the case of the first three harmonic terms varies only from 6·9 per cent. to 6·6 per cent. of the radius, and the shielding factor from ·058 to ·055.

If there are three concentric shells, by differentiating the expression for Ψ in equation (3) we get

$$\left.\begin{aligned}
&\frac{\mathrm{D}^2 d\Psi}{\psi_0\mu^3(\mathrm{N}+1)^6\alpha_0\alpha_1\alpha_2\alpha_3\alpha_4}\\
&\quad=\left[\xi\eta\alpha_1\alpha_4(\alpha_4-\alpha_5)\{\eta\alpha_1(\alpha_2-\alpha_3)-\alpha_2(\eta\alpha_2-\xi\alpha_3)\}\right.\\
&\qquad\qquad\left.+\eta\alpha_1\alpha_3(\xi\alpha_4-\eta\alpha_5)\{\xi\alpha_2(\alpha_2-\alpha_3)-\alpha_1(\xi\alpha_2-\eta\alpha_3)\}\right]\frac{d\alpha_0}{\alpha_0}\\
&\quad+\left[\eta\alpha_3(\xi\alpha_4-\eta\alpha_5)\{(\xi\alpha_2-\eta\alpha_3)\alpha_1^2-\xi\alpha_0\alpha_2(\alpha_2-\alpha_3)\}\right.\\
&\qquad\qquad\left.+\xi\eta\alpha_4(\alpha_4-\alpha_5)\{(\eta\alpha_2-\xi\alpha_3)\alpha_0\alpha_2-\eta\alpha_1^2(\alpha_2-\alpha_3)\}\right]\frac{d\alpha_1}{\alpha_1}\\
&\quad+\left[\eta\alpha_3(\xi\alpha_4-\eta\alpha_5)\{-(\xi\alpha_0-\eta\alpha_1)\alpha_1\alpha_3+\xi\alpha_2^2(\alpha_0-\alpha_1)\}\right.\\
&\qquad\qquad\left.+\xi\eta\alpha_4(\alpha_4-\alpha_5)\{(\xi\alpha_0-\eta\alpha_1)\alpha_1\alpha_3-\eta\alpha_2^2(\alpha_0-\alpha_1)\}\right]\frac{d\alpha_2}{\alpha_2}\\
&\quad+\left[\eta\alpha_3^2(\xi\alpha_4+\eta\alpha_5)\{\alpha_1(\xi\alpha_0-\eta\alpha_1)-\xi\alpha_2(\alpha_0-\alpha_1)\}\right.\\
&\qquad\qquad\left.+\xi\eta\alpha_2\alpha_4(\alpha_4-\alpha_5)\{\eta\alpha_2(\alpha_0-\alpha_1)-\alpha_1(\xi\alpha_0-\eta\alpha_1)\}\right]\frac{d\alpha_3}{\alpha_3}\\
&\quad+\left[\eta\alpha_1(\xi\alpha_0-\eta\alpha_1)\{\xi\alpha_4^2(\alpha_2-\alpha_3)-\alpha_3\alpha_5(\xi\alpha_2-\eta\alpha_3)\}\right.\\
&\qquad\qquad\left.+\xi\eta\alpha_2(\alpha_0-\alpha_1)\{(\xi\alpha_3-\eta\alpha_2)\alpha_4^2+\eta\alpha_3\alpha_5(\alpha_2-\alpha_3)\}\right]\frac{d\alpha_4}{\alpha_4}\\
&\quad+\left[\eta\alpha_1(\xi\alpha_0-\eta\alpha_1)\{\xi\alpha_4(\alpha_2-\alpha_3)+\alpha_3(\xi\alpha_2-\eta\alpha_3)\}\right.\\
&\qquad\qquad\left.+\xi\eta\alpha_2(\alpha_0-\alpha_1)\{(\eta\alpha_2-\xi\alpha_3)\alpha_4-\eta(\alpha_2-\alpha_3)\alpha_3\}\right]d\alpha_5.
\end{aligned}\right\}\quad(10)$$

We note that in this, as in the previous case, there can be no maximum or minimum if α_0 or α_5 are independent variables.

Further, writing the coefficient of $d\alpha_0$ in the form

$$\eta\alpha_1\alpha_4(\xi\alpha_4-\xi\alpha_5)\{\eta\alpha_1(\alpha_2-\alpha_3)-\alpha_2(\eta\alpha_2-\xi\alpha_3)\}$$
$$-\eta\alpha_1\alpha_3(\xi\alpha_4-\eta\alpha_5)\{\alpha_1(\xi\alpha_2-\eta\alpha_3)-\xi\alpha_2(\alpha_2-\alpha_3)\},$$

if we remember that the α's diminish from α_1 to α_5 and that ξ is $> \eta$, it is easy to see that the coefficient is the difference of two positive quantities of which the second is the larger.

For

$$\xi\alpha_4-\eta\alpha_5>\xi\alpha_4-\xi\alpha_5$$

and

$$\{\alpha_1(\xi\alpha_2-\eta\alpha_3)-\xi\alpha_2(\alpha_2-\alpha_3)-\{\eta\alpha_1(\alpha_2-\alpha_3)-\alpha_2(\eta\alpha_2-\xi\alpha_3)\}$$
$$=(\xi-\eta)\alpha_2(\alpha_1-\alpha_2),$$

which is positive.

Hence the whole coefficient is negative, or Ψ diminishes as α_0 increases, *i.e.* as the radius a_0 diminishes. Thus, as we should expect, we obtain the same result as before, viz., that the reduction of the radius of the innermost surface always improves the shielding. In like manner if a_5 increases the shielding improves.

Let us now suppose, as before, that the outermost and innermost radii are given. The best arrangement is then found by equating the coefficients of $d\alpha_1$, $d\alpha_2$, and $d\alpha_3$, and $d\alpha_4$ to zero.

By eliminating the terms in α_4 and α_5 from the first pair, and also from the second pair of these equations, we get

$$\alpha_2\alpha_3(\alpha_1-\alpha_2)\{\xi\alpha_0(\alpha_0-\alpha_1)\alpha_2-\alpha_1^2(\xi\alpha_0-\eta\alpha_1)\}=0 \quad . \quad (11)$$

and

$$\alpha_3(\alpha_2-\alpha_3)\{\xi\eta(\alpha_0-\alpha_1)^2\alpha_2^3+(\xi\alpha_0-\eta\alpha_1)^2\alpha_1^2\alpha_3$$
$$-\xi(\xi\alpha_0-\eta\alpha_1)\alpha_1\alpha_2(\alpha_2+\alpha_3)(\alpha_0-\alpha_1)=0 \quad . \quad (12)$$

In the cross multiplications by which these equations are obtained, we assume $\alpha_4-\alpha_5\neq0$.

If $\alpha_1=\alpha_2$ or $\alpha_2=\alpha_3$ two of the shells are fused into one. Putting this case aside and substituting for $(\xi\alpha_0-\eta\alpha_1)/(\alpha_0-\alpha_1)$ in the second equation from the first we get

$$\alpha_1\alpha_2(\xi\alpha_0-\eta\alpha_1)-\xi\alpha_0\alpha_3(\alpha_0-\alpha_1)=0.$$

Hence and from (11),

$$\frac{\xi\alpha_0-\eta\alpha_1}{\xi\alpha_0(\alpha_0-\alpha_1)}=\frac{\alpha_3}{\alpha_1\alpha_2}=\frac{\alpha_2}{\alpha_1^2}, \quad . \quad . \quad . \quad . \quad (13)$$

$$\therefore \quad \alpha_3\alpha_1=\alpha_2^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Again equating the coefficient of $d\alpha_3$ to zero, substituting for $(\xi\alpha_0-\eta\alpha_1)/(\alpha_0-\alpha_1)$, rearranging the terms, and substituting again, we get

$$\frac{\xi\alpha_4-\eta\alpha_5}{\xi\alpha_4(\alpha_4-\alpha_5)}=\frac{\alpha_0\alpha_2^2}{\alpha_1^2\alpha_3^2}=\frac{\alpha_0}{\alpha_2^2} \text{ by (14)}, \quad . \quad . \quad (15)$$

Next, equating the coefficient of $d\alpha_4$ to zero, we get

$$\frac{\xi\alpha_4^2-\eta\alpha_3\alpha_5}{\alpha_4^2-\alpha_3\alpha_5}=\frac{\alpha_2}{\alpha_3}\times\frac{\xi\alpha_0-\eta\alpha_1}{\alpha_0-\alpha_1}=\frac{\xi\alpha_0}{\alpha_1}.$$

This again reduces to

$$\frac{\alpha_4^2}{\alpha_3\alpha_5}=\frac{\xi\alpha_0-\eta\alpha_1}{\xi(\alpha_0-\alpha_1)}=\frac{\alpha_0\alpha_2}{\alpha_1^2} \text{ by (13)}, \quad . \quad . \quad (16)$$

Again, since $\alpha_2^2=\alpha_3\alpha_1$,

$$\therefore \quad \alpha_3\alpha_1(\xi\alpha_4-\eta\alpha_5)=\xi\alpha_0\alpha_4(\alpha_4-\alpha_5). \quad . \quad . \quad . \quad . \quad (17)$$

But

$$\alpha_3\alpha_5(\xi\alpha_0-\eta\alpha_1)=\xi\alpha_4^2(\alpha_0-\alpha_1),$$

or

$$\xi\alpha_0(\alpha_3\alpha_5-\alpha_4^2)=\alpha_1(\eta\alpha_3\alpha_5-\xi\alpha_4^2).$$

Multiplying this by (17) and putting $\alpha_2^2=\alpha_3\alpha_1$,

$$\alpha_3(\xi\alpha_4-\eta\alpha_5)(\alpha_3\alpha_5-\alpha_4^2)=\alpha_4(\alpha_4-\alpha_5)(\eta\alpha_3\alpha_5-\xi\alpha_4^2),$$

$$\therefore (\alpha_3-\alpha_4)\{\alpha_3\alpha_5(\xi\alpha_4-\eta\alpha_5)-\xi\alpha_4^2(\alpha_4-\alpha_5)\}=0.$$

If $\alpha_3=\alpha_4$ the problem is reduced to the case of two shells. If $\alpha_3\neq\alpha_4$,

$$\frac{\xi\alpha_4-\eta\alpha_5}{\xi(\alpha_4-\alpha_5)}=\frac{\alpha_4^2}{\alpha_3\alpha_5}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Hence the equations (13), (15), (16), and (18) finally reduce to

$$\frac{\xi\alpha_0-\eta\alpha_1}{\xi(\alpha_0-\alpha_1)}=\frac{\alpha_3\alpha_0}{\alpha_1\alpha_2}=\frac{\alpha_2\alpha_0}{\alpha_1^2}=\frac{\alpha_4^2}{\alpha_3\alpha_5}=\frac{\alpha_0\alpha_4}{\alpha_2^2}. \quad . \quad . \quad (19)$$

It must be remembered that in obtaining these equations we have thrown out the alternative cases where any two consecutive values of α become identical; *i. e.* cases in which the three shells become two, either by fusion or by the disappearance of one.

Thus the solution for the case of two shells is contained in the general equations for three shells, but the equality of the above groups of ratios holds good only when there are three.

It remains to assure ourselves that in the case of three shells all the variables give minima simultaneously.

Thus, considering the coefficient of $d\alpha_1$ the coefficient of α_1^2 is

$$\eta\alpha_3(\xi\alpha_4-\eta\alpha_5)(\xi\alpha_2-\eta\alpha_3)$$
$$-\eta\alpha_4(\xi\alpha_4-\xi\alpha_5)(\eta\alpha_2-\eta\alpha_3).$$

Comparing these term by term, we see that the terms are either equal or are less in the second expression. Hence the coefficient of α_1^2 is positive. If, then, α_1 is a little less than the critical value, the coefficient of $d\alpha_1$ must be negative. Hence, as α_1 increases up to the critical value Ψ diminishes, *i. e.* the critical value of α_1 corresponds to a minimum.

The other coefficients if treated in the same way lead to the same conclusion. We may now proceed to put equation (19) into the simplest form for calculation.

Since

$$\frac{\alpha_2}{\alpha_3}=\frac{\alpha_1}{\alpha_2},$$

$$\therefore\quad \frac{\alpha_4}{\alpha_3\alpha_5}=\frac{\alpha_0}{\alpha_2^{\,2}}=\frac{\alpha_0}{\alpha_1\alpha_3},\quad \therefore\ \frac{\alpha_4}{\alpha_5}=\frac{\alpha_0}{\alpha_1},$$

and since

$$\frac{\alpha_2\alpha_0}{\alpha_1^{\,2}}=\frac{\alpha_4^{\,2}}{\alpha_3\alpha_5},\quad \therefore\ \frac{\alpha_2}{\alpha_1}=\frac{\alpha_4}{\alpha_3};$$

so that we have three independent equations, viz.,

$$\left.\begin{array}{l}\dfrac{\alpha_0}{\alpha_1}=\dfrac{\alpha_4}{\alpha_5}=\lambda \text{ say,}\\ \text{and} \\ \dfrac{\alpha_1}{\alpha_2}=\dfrac{\alpha_2}{\alpha_3}=\dfrac{\alpha_3}{\alpha_4}=\lambda' \text{ say,}\end{array}\right\}\ldots \quad (20)$$

together with

$$(\xi\lambda-\eta)\lambda'=\xi(\lambda-1)\lambda\,;\ \ldots\ldots \quad (21)$$

also

$$\frac{\alpha_0}{\alpha_5}=\lambda^2\lambda'^3=\text{L say as before.}$$

So that equation (21) becomes

$$(\xi\lambda-\eta)\,L^{\frac{1}{3}}=\xi(\lambda-1)\lambda^{\frac{5}{3}}$$

or

$$(\lambda-1+\epsilon)\,L^{\frac{1}{3}}=(\lambda-1)\lambda^{\frac{5}{3}}. \quad . \quad . \quad . \quad . \quad (22)$$

If $\lambda'=1$ all the shells fuse into one (without passing through the intermediate stage of two shells). In this case $\lambda^2=L$.

Substituting in (22) we get

$$(\sqrt{L}-1)^2=\epsilon,$$

which is the same as (9).

Thus the ratio of the outermost to the innermost radius below is the same for two and for three shells.

The maxima of advantage in the two cases are independent, and to determine between them their magnitudes must be compared.

Substituting in (3) from equation (20) and simplifying by means of (22), we finally get

$$\frac{\Psi}{\psi_0}=\frac{\epsilon^3 L^{\frac{4}{3}}}{(\lambda-1)^4\{\lambda^{\frac{5}{3}}-L^{\frac{1}{3}}(1-\epsilon)\}}, \quad . \quad . \quad (23)$$

where λ is given by equation (22).

In the limiting case when $\lambda^2=L=(1+\sqrt{\epsilon})^2$,

$$\Psi/\psi_0=\frac{\sqrt{\epsilon}(1+\sqrt{\epsilon})}{2},$$

which is, as it should be, the same as the value obtained by inserting the same values of λ and L in the corresponding expression for two shells. When ϵ is negligible as compared with λ and L we get from (22) $\lambda^5=L$, and from (23)

$$\frac{\Psi}{\psi_0}=\frac{\epsilon^3 L}{(\sqrt[5]{L}-1)^5}=\frac{\epsilon^3\lambda^5}{(\lambda-1)^5}=\frac{(N+1)^6}{N^3\mu^3}\times\frac{L}{(\sqrt[5]{L}-1)^5}.$$

If we take the numbers before employed, viz. :—

$$\mu=501, \quad a_0=8a_3, \quad N=2,$$

we have $\lambda=\sqrt[5]{8}=1\cdot516$;

$$\therefore \quad \frac{\Psi}{\psi_0}=\frac{3^6}{500^3}\times\left(\frac{1}{0\cdot516}\right)^5=0\cdot00016.$$

The results so far obtained in the special case to which numerical calculation has been applied may be summed up as follows, the arrangement being the best in each case :—

	Volume of Material used.	External Field.
Single shell	1·0	0·018
Two shells...............	5·0	0·0006
Three shells	4·8	0·00016
Single shell	7·0	0·0102

The great advantage of the lamination of the shielding material is here well exhibited.

As the external field varies slowly in the neighbourhood of the minimum, a considerable economy of material may be effected without much sacrifice of efficiency. Thus, in the case of two shells, we have seen that whereas to secure the minimum external field (0·0006 of the unshielded field) the volume of the material used must (in the case considered) be five times the internal volume, a practically identical result (0·0007) can be obtained by the use of a volume 3·66, while an efficiency four times greater can be obtained by a volume 4·8.

Best arrangement in the form of Two Shells separated by an Air-gap when the innermost radius and the volume of the material are given.

So far I have supposed that both the innermost and outermost radii are given.

It is evident that we may reduce the number of the independent variables, and I now proceed to discuss two problems which might occur in practice. The enunciation of the first of these is given above.

In neither case is the solution very simple, and I therefore confine myself to the case of two shells when the magnet is in the centre. In this case $\alpha_1 = 1/a_1^3$ &c., so that the condition that the volume shall be constant is

$$\frac{1}{\alpha_3} - \frac{1}{\alpha_2} + \frac{1}{\alpha_1} - \frac{1}{\alpha_0} = \frac{1}{\mathrm{R}} \text{ say.}$$

Hence

$$\frac{d\alpha_3}{\alpha_3^2} - \frac{d\alpha_2}{\alpha_2^2} + \frac{d\alpha_1}{\alpha_1^2} = 0.$$

Substituting for $d\alpha_2$ in equation (5), and remembering that $d\alpha_0=0$, we get as the conditions for a critical value

$$\xi\alpha_2(\alpha_0-\alpha_1)\{\alpha_2^2-\alpha_3^2\}-\alpha_1\alpha_3(\alpha_2-\alpha_3)(\xi\alpha_0-\eta\alpha_1) = 0 \quad . \quad (24)$$

and

$$\{-\xi\eta\alpha_1\alpha_0(\alpha_2-\alpha_3)+\eta\alpha_1^2(\xi\alpha_2-\eta\alpha_3)\}\alpha_1$$
$$+\{\xi\eta\alpha_2^2(\alpha_0-\alpha_1)-\eta\alpha_1\alpha_3(\xi\alpha_0-\eta\alpha_1)\}\alpha_2=0. \quad . \quad . \quad (25)$$

From the first of these we get

$$(\xi\alpha_0-\eta\alpha_1)\alpha_1\alpha_3 = \xi\alpha_2(\alpha_0-\alpha_1)(\alpha_2+\alpha_3). \quad . \quad . \quad . \quad (26)$$

Substituting for $\xi\alpha_0-\eta\alpha_1$ in (25),

$$(\xi\alpha_2-\eta\alpha_3)\alpha_1^3 = \xi\alpha_2\{\alpha_0\alpha_1(\alpha_2-\alpha_3)+\alpha_2\alpha_3(\alpha_0-\alpha_1)\},$$

multiply by α_1 and subtract,

$$\therefore \quad \alpha_1^2(\alpha_0\alpha_3-\alpha_1\alpha_2) = \alpha_2\{-\alpha_0\alpha_1(\alpha_2-\alpha_3)$$
$$+(\alpha_0-\alpha_1)(\alpha_1\alpha_2+\alpha_1\alpha_3-\alpha_2\alpha_3)\};$$

$$\therefore \quad \alpha_1(\alpha_0\alpha_3-\alpha_1\alpha_2) = \alpha_2\alpha_3(\alpha_0-\alpha_1);$$

or

$$\alpha_0\alpha_3(\alpha_1-\alpha_2) = \alpha_1\alpha_2(\alpha_1-\alpha_3). \quad . \quad . \quad . \quad . \quad (27)$$

But

$$\frac{1}{\alpha_3}-\frac{1}{\alpha_2}+\frac{1}{\alpha_1}-\frac{1}{\alpha_0} = \frac{1}{\mathrm{R}},$$

or

$$(\alpha_0-\alpha_3)\alpha_2\alpha_1-(\alpha_1-\alpha_2)\alpha_0\alpha_3 = \frac{\alpha_0\alpha_1\alpha_2\alpha_3}{\mathrm{R}};$$

∴ by (27),

$$\alpha_2\alpha_1\{\alpha_0-\alpha_3-(\alpha_1-\alpha_3)\} = \frac{\alpha_0\alpha_1\alpha_2\alpha_3}{\mathrm{R}};$$

$$\therefore \quad \alpha_0-\alpha_1 = \frac{\alpha_0\alpha_3}{\mathrm{R}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Hence

$$\alpha_1 = \alpha_0\left(1-\frac{\alpha_3}{\mathrm{R}}\right).$$

By combining this with (27),

$$\alpha_2\left\{\alpha_0\left(1-\frac{\alpha_3}{\mathrm{R}}\right)^2+\frac{\alpha_3^2}{\mathrm{R}}\right\} = \alpha_0\alpha_3\left(1-\frac{\alpha_3}{\mathrm{R}}\right); \quad . \quad (29)$$

also from (26) and (28),

$$(\xi\alpha_0-\eta\alpha_1)\alpha_1=\frac{\xi\alpha_2\alpha_0}{R}(\alpha_2+\alpha_3),$$

and substituting for α_2 from (29), and dividing by $1-\frac{\alpha_3}{R}$, this becomes

$$\left\{\xi-\eta+\frac{\eta\alpha_3}{R}\right\}\left\{\alpha_0^2\left(1-\frac{\alpha_3}{R}\right)^3+\frac{2\alpha_0\alpha_3^2}{R}\left(1-\frac{\alpha_3}{R}\right)\right\}$$
$$-\frac{\xi}{R}\alpha_3^2\alpha_0\left\{2-\frac{\alpha_3}{R}\right\}-\frac{\eta\alpha_3^4}{R^2}=0. \quad . \quad . \quad . \quad (30)$$

The coefficient of α_3^4 in this expression is negative when $\alpha_3=-\infty$.

It is positive when $\alpha_3=0$, and negative when $\alpha_3=\infty$.

Hence the equation has a positive root. In order that the solution may be applicable, this root must be such that

$$\frac{1}{\alpha_3}-\frac{1}{\alpha_0}>\frac{1}{R};$$

i. e. the space enclosed between the inner boundary of the inner shell and the outer boundary of the outer one must be greater than the volume of the material used. The change in the sign of the expression from positive to negative as α_3 increases must therefore take place when

$$\frac{1}{\alpha_3}>\frac{1}{\alpha_0}+\frac{1}{R},$$

or the expression must be negative when

$$\frac{1}{\alpha_3}=\frac{1}{\alpha_0}+\frac{1}{R}.$$

Now equation (30) may be thrown into the form

$$\left(\frac{\xi-\eta}{\alpha_3}+\frac{\eta}{R}\right)\left\{\alpha_0^2\left(\frac{1}{\alpha_3}-\frac{1}{R}\right)^3+\frac{2\alpha_0}{R}\left(\frac{1}{\alpha_3}-\frac{1}{R}\right)\right\}$$
$$-\frac{\xi}{R}\frac{\alpha_0}{\alpha_3}\left(\frac{2}{\alpha_3}-\frac{1}{R}\right)-\frac{\eta}{R^2}=0;$$

and putting

$$\frac{1}{\alpha_3}-\frac{1}{R}=\frac{1}{\alpha_0},$$

the left-hand side becomes

$$\xi\left(\frac{1}{\mathrm{R}}+\frac{1}{\alpha_0}\right)^2\left\{\frac{\xi-\eta}{\xi}-\frac{\alpha_0}{\mathrm{R}}\right\}.$$

Hence the condition that there shall be an arrangement which gives a minimum external field is that

$$\frac{\alpha_0}{\mathrm{R}}>\frac{\xi-\eta}{\xi},\quad \textit{i.e.} >\epsilon,\quad \textit{i.e.} >\frac{9\mu}{(2\mu+1)(2+\mu)}.$$

If we confine ourselves to the case of a small magnet in the centre of the shell: in the limiting case

$$\mathrm{L}=\frac{\alpha_0}{\alpha_3}=1+\frac{\alpha_0}{\mathrm{R}}=1+\epsilon,$$

as against

$$\mathrm{L}=(1+\sqrt{\epsilon})^2,$$

obtained when the internal and external radii were given.

If μ is nearly equal to unity, this reduces to

$$\frac{1}{\mathrm{R}}>\frac{1}{\alpha_0};$$

i.e. there will be no "best arrangement" unless the volume of the shielding material exceeds the volume of the space enclosed. If μ is very large, the expression becomes

$$\frac{1}{\mathrm{R}}>\frac{1}{\alpha_0}\times\frac{9}{2\mu}.$$

Thus, if $\mu = 500$, $\qquad \dfrac{1}{\mathrm{R}}>\dfrac{0{\cdot}009}{\alpha_0}.$

There will be no "best arrangement" unless the volume of the shielding material is greater than (in round numbers) one hundreth of the volume enclosed; *i.e.* unless it is more than sufficient to form a shell the thickness of which is $=0{\cdot}003$ of the radius.

In this case therefore, as in that in which the radius of the outer surface was given, if very thin shells are used, lamination is injurious. The higher the permeability the smaller is the lower limit of thickness above which lamination is useful. If the permeability is great, the limit is $\dfrac{3}{2\mu}\times$ radius. If we put

$$\frac{1}{\alpha_3}=\frac{1}{\mathrm{R}}+\frac{1}{\alpha_0}$$

in equation (29), we get $\alpha_2=\alpha_3$, *i.e.* the thickness of the external shell vanishes in the limiting case.

I now proceed to give some numerical examples of the application of these formulæ. Equation (30) is troublesome, but when μ is large so that ξ and η are nearly equal it may be simplified.

It then reduces to

$$\alpha_0^2\left(\frac{1}{\alpha_3}-\frac{1}{\mathrm{R}}\right)^3+\frac{2\alpha_0}{\mathrm{R}}\left(\frac{1}{\alpha_3}-\frac{1}{\mathrm{R}}\right)-\frac{\alpha_0}{\alpha_3}\left(\frac{2}{\alpha_3}-\frac{1}{\mathrm{R}}\right)-\frac{1}{\mathrm{R}}=0. \quad . \quad . \quad (31)$$

We have already seen that when the radius of the outer shell is twice the smallest radius, the best result is obtained when the volume of the shells is five times the volume of the space enclosed by the smaller shell.

The above equation may be used to determine whether a still better arrangement is possible with the same volume of material, if we suppose the radius of the external surface variable.

If $\alpha_0=1$, $1/\mathrm{R}=5$, and equation (31) is found to have a root such that

$$1/\alpha_3 = 9{\cdot}5 \text{ nearly.}$$

This gives

$$\alpha_3 = \sqrt[3]{9{\cdot}5} = 2{\cdot}118\,;$$

and, as in the previous calculation we assumed $\alpha_3=2$, it is evident that the two critical points nearly coincide.

In this case we get

$$\alpha_1 = \alpha_0(1-\alpha_3/\mathrm{R}) = 9/19,$$

from

$$\alpha_2 = 18/101.$$

The magnitude of the shielded field is practically the same as before.

It must be remembered that neither the equation in α_3 nor the ratio Ψ/ψ_0 are valid when ξ is put $=\eta$ if the shells are thin or (which amounts to the same thing) if R is large.

The best method of dealing with the problem is, then, as follows:—

Write $$\xi = \eta(1+\epsilon),$$

where ϵ is so small that terms in ϵ may be neglected, unless divided by a small quantity.

Then equation (30) may be written

$$\left(1+\frac{\epsilon \mathrm{R}}{\alpha_3}\right)\left\{\alpha_0^3\left(\frac{1}{\alpha_3}-\frac{1}{\mathrm{R}}\right)^3+\frac{2\alpha_0}{\mathrm{R}}\left(\frac{1}{\alpha_3}-\frac{1}{\mathrm{R}}\right)\right\}$$

$$-\frac{\alpha_0}{\alpha_3}\left(\frac{2}{\alpha_3}-\frac{1}{\mathrm{R}}\right)(1+\epsilon)-\frac{1}{\mathrm{R}}=0.$$

The terms in ϵ may therefore be neglected unless R/α_3 is large.

As an example, let $\mathrm{R}=10$, *i.e.* let the volume of the metal be one tenth of the volume of the enclosure.

If then, as before, we take $\mu=501$, $\alpha_0=1$,

$$\therefore \quad \xi=\eta(1+0{\cdot}009) \quad \text{and} \quad \epsilon=0{\cdot}009.$$

If we neglect all the terms in ϵ we get

$$\frac{1}{\alpha_3}=2{\cdot}17.$$

Hence $$\epsilon \mathrm{R}/\alpha_3 = {\cdot}1953,$$

which cannot be neglected when compared with unity.

Taking this term into account,

$$1/\alpha_3 = 1{\cdot}875.$$

As the radii vary inversely as the cube roots of the α's, the two values of the external radius are 1·233 and 1·294. As the shielding varies slowly when near a minimum value, as the outer shell is displaced outward, it is probable that no very grave error would be introduced even in this case by the use of the approximate equation. The error would, however, increase rapidly as the volume of material employed diminished.

Using the more correct value we get

$$\alpha_1=\alpha_0\left(1-\frac{\alpha_3}{\mathrm{R}}\right)=1-0{\cdot}0533=0{\cdot}9467,$$

$$-\frac{1}{\alpha_2}=\frac{1}{\mathrm{R}}-\frac{1}{\alpha_3}-\frac{1}{\alpha_1}+\frac{1}{\alpha_0}=0{\cdot}1-1{\cdot}875-1{\cdot}056+1=-1{\cdot}831.$$

Hence we have for the best arrangement,

$$\alpha_3=0{\cdot}5333, \quad \alpha_2=0{\cdot}5461, \quad \alpha_1=0{\cdot}9467, \quad \alpha_0=1,$$

or $$\alpha_3=1{\cdot}233, \quad \alpha_2=1{\cdot}223, \quad \alpha_1=1{\cdot}018, \quad \alpha_0=1.$$

It will be convenient to throw equation (4) into a form suitable for calculation when the shells are thin and the permeability is great.

With the same notation as that just employed it becomes

$$\Psi(\alpha_0-\alpha_1)(\alpha_2-\alpha_3)\left\{-\alpha_2+\alpha_1\left(1+\frac{\alpha_3\epsilon}{\alpha_2-\alpha_3}\right)\left(1+\frac{\alpha_1\epsilon}{\alpha_0-\alpha_1}\right)\right\}$$
$$=\frac{\psi_0\alpha_0\alpha_1\alpha_2(\mathrm{N}+1)^4}{\mathrm{N}^2\mu^2},$$

which, in the case under consideration, leads to the result

$$\Psi/\psi_0=0{\cdot}064.$$

By (2), if the same amount of material were concentrated in a single shell, we should have

$$\Psi/\psi_0=0{\cdot}089.$$

In this case therefore the advantage of the best arrangement is much less than if a larger amount of material were used.

No very great disadvantage is entailed in such a case if the outer shell be outside its best position, as, if we take $1/\alpha_3=3$, we get

$$\Psi/\psi_0=0{\cdot}069,$$

which does not differ very much from the minimum value.

Case of Two Shells when the largest and smallest radii and the volume of the material used are given.

In this case $d\alpha_0=d\alpha_3=0$, and as before

$$\frac{1}{\alpha_3}-\frac{1}{\alpha_2}+\frac{1}{\alpha_1}-\frac{1}{\alpha_0}=\frac{1}{\mathrm{R}},$$
$$\frac{1}{\alpha_2}-\frac{1}{\alpha_1}=\frac{1}{\rho},$$

where, since α_3 and α_0 are given, ρ is a known quantity.

Hence substituting from

$$\frac{d\alpha_2}{\alpha_2{}^2}=\frac{d\alpha_1}{\alpha_1{}^2}\text{ in (5)}$$

we get

$$[\xi\eta\{\alpha_1\alpha_2(\alpha_1+\alpha_2)-\alpha_0\alpha_2{}^2\}-\eta^2\alpha_1{}^2\alpha_3]\,(\alpha_2-\alpha_3)=0,$$

or

$$\alpha_2{}^2(\alpha_0-\alpha_1)-\alpha_2\alpha_1{}^2=-\frac{\eta}{\xi}\alpha_1{}^2\alpha_3.$$

But
$$\frac{1}{a_2}=\frac{\rho+a_1}{\rho a_1},$$

$$\therefore\quad \frac{\rho^2(a_0-a_1)}{(\rho+a_1)^2}-\frac{\rho a_1}{\rho+a_1}=-\frac{\eta a_3}{\xi},$$

or
$$(\rho+a_1)^2=\frac{\xi\rho^2(a_0+\rho)}{\xi\rho-\eta a_3},$$

whence a_1 is known.

In the case when μ is so great that ξ may be taken $=\eta$, this becomes

$$(\rho+a_1)^2(\rho-a_3)=\rho^2(a_0+\rho).$$

If the given volume is very nearly $=\frac{4}{3}\pi(a_3^3-a_0^3)$, $1/\rho$ is small.

In this case the solution reduces to

$$2a_1\xi-\eta a_3=\xi a_0,$$

or
$$a_1=\frac{a_3\eta+\xi a_0}{2\xi}.$$

If μ is large, this becomes

$$a_1=\frac{a_3+a_0}{2}\qquad\text{or}\qquad\frac{1}{a_1^{3}}=\frac{1}{2}\left(\frac{1}{a_3^{2}}+\frac{1}{a_0^{2}}\right).$$

Hence, if the external and internal radii are given, the best position for a narrow crack within the shell is such that the volume it encloses is the harmonic mean of the volumes enclosed by the inner and outer surfaces of the shell respectively.

Case of Two contiguous Shells of different Permeabilities.

It is evident that the question as to the best arrangement of two shells of different permeabilities cannot be very definitely answered.

If the dimensions of the shells are variable, the more of the more permeable material we use, the more complete will be the shielding.

If, however, the dimensions of the shells are fixed, under certain circumstances (such as when its thickness is much the greater) it will be better to make the outer shell of the more permeable material, while under other conditions the reverse arrangement may be best.

Though the problem is not of any great practical interest, it may be worth while to indicate the mode of solution.

Let $\mu_0 = \mu_3 = 1$, and let the permeabilities of the two shells be μ and μ_2. Then we get

$$\begin{aligned}\Psi\{ &-\alpha_1\alpha_2 N(N\mu_1+\mu_2)(\mu_2-1)(\mu_1-1)-\alpha_1^2 N(N+\mu_2)(\mu_1-\mu_2)(\mu_1-1)\\ &+\alpha_0\alpha_2(N\mu_1+1)N(\mu_2-1)(\mu_1-\mu_2)+\alpha_0\alpha_1(N\mu_1+1)(N+\mu_2)(N\mu_2+\mu_1)\\ &=\psi_0\alpha_0\alpha_1(N+1)^3\mu_1\mu_2.\end{aligned}$$

Hence, if Ψ_2 and Ψ_1 be the external potential when the permeability of the external shell is μ_2 and μ_1 respectively, we get, by exchanging the positions of the symbols μ_1 and μ_2 in the above expression and subtracting,

$$\begin{aligned}\psi\alpha_0\alpha_1(N+1)^3\mu_1\mu_2&\left\{\frac{1}{\Psi_2}-\frac{1}{\Psi_1}\right\}\\ &=(\mu_1-\mu_2)\big[-\alpha_1^2 N\{2\mu_1\mu_2+(N-1)(\mu_1+\mu_2)-2N\}\\ &\qquad -\alpha_1(\alpha_2+\alpha_0)N(N-1)(\mu_2-1)(\mu_1-1)\\ &\qquad +\alpha_0\alpha_2 N\{2N\mu_1\mu_2-(N-1)(\mu_1+\mu_2)-2\}\big].\end{aligned}$$

Hence the exchange of materials causes no difference if either $\mu_1 = \mu_2$ or if the second factor on the right-hand side vanishes.

If both the shells are thin let $a_2 = a_1 + t_1$, and $a_0 = a_1 - t_0$, where the squares of t_1 and t_2 may be neglected.

Then

$$\alpha_2 = (a_1+t_2)^{-(2n+1)} = \alpha_1\left\{1-(2n+1)\frac{t_2}{a_1}\right\}$$

$$\alpha_0 = (a_1-t_0)^{-(2n+1)} = \alpha_1\left\{1+(2n+1)\frac{t_0}{a_1}\right\}.$$

Substituting these approximate values, the second factor becomes

$$-(2n+1)N(N+1)(\mu_1\mu_2-1)(t_2-t_0)/a_1,$$

which proves that, as we should expect, if the shells are thin the order of their arrangement is indifferent.

Case of Three Shells of different Permeabilities.

The general relation between Ψ and ψ_0 when an internal space of permeability μ_0 is surrounded by three concentric spherical shells of permeabilities μ_1, μ_2, μ_3, and when the permeability of external space is μ_4, is

$$
\begin{aligned}
&\psi_0\alpha_0\alpha_1\alpha_2 \,.\, \mu_0\mu_1\mu_2\mu_3(\mathrm{N}+1)^4/\Psi \\
&\quad=\alpha_1\alpha_2\alpha_3\mathrm{N}(\mu_3-\mu_4)(\mu_0-\mu_1)(\mathrm{N}\mu_2+\mu_3)(\mathrm{N}\mu_1+\mu_2) \\
&\qquad+\alpha_1\alpha_2^2\mathrm{N}(\mathrm{N}\mu_4+\mu_3)(\mu_2-\mu_3)(\mu_0-\mu_1)(\mathrm{N}\mu_1+\mu_2) \\
&\qquad+\alpha_1^2\alpha_3\mathrm{N}^2(\mu_3-\mu_4)(\mu_2-\mu_3)(\mu_1-\mu_2)(\mu_0-\mu_1) \\
&\qquad+\alpha_1^2\alpha_2\mathrm{N}(\mathrm{N}\mu_4+\mu_3)(\mathrm{N}\mu_3+\mu_2)(\mu_1-\mu_2)(\mu_0-\mu_1) \\
&\qquad+\alpha_0\alpha_2\alpha_3\mathrm{N}(\mathrm{N}\mu_2+\mu_3)(\mathrm{N}\mu_1+\mu_0)(\mu_3-\mu_4)(\mu_1-\mu_2) \\
&\qquad+\alpha_0\alpha_2^2\mathrm{N}(\mathrm{N}\mu_4+\mu_3)(\mathrm{N}\mu_1+\mu_0)(\mu_2-\mu_3)(\mu_1-\mu_2) \\
&\qquad+\alpha_0\alpha_1\alpha_3\mathrm{N}(\mathrm{N}\mu_2+\mu_1)(\mathrm{N}\mu_1+\mu_0)(\mu_3-\mu_4)(\mu_2-\mu_3) \\
&\qquad+\alpha_0\alpha_1\alpha_2(\mathrm{N}\mu_1+\mu_0)(\mathrm{N}\mu_2+\mu_1)(\mathrm{N}\mu_3+\mu_2)(\mathrm{N}\mu_4+\mu_3).
\end{aligned}
$$

If now we put $\mu_0=\mu_2=\mu_4=1$, this reduces to the case of two concentric shells of permeabilities μ_1 and μ_3 respectively in air and separated by an air-gap. The equation then becomes

$$
\begin{aligned}
&\psi_0\alpha_0\alpha_1\alpha_2 \,.\, \mu_1\mu_3(\mathrm{N}+1)^4/\Psi \\
&\quad=-\alpha_1\alpha_2\alpha_3\mathrm{N}(\mu_3-1)(\mu_1-1)(\mathrm{N}+\mu_3)(\mathrm{N}\mu_1+1) \\
&\qquad+\alpha_1\alpha_2^2\mathrm{N}(\mathrm{N}+\mu_3)(\mu_3-1)(\mu_1-1)(\mathrm{N}\mu_1+1) \\
&\qquad+\alpha_1^2\alpha_3\mathrm{N}^2(\mu_3-1)^2(\mu_1-1)^2 \\
&\qquad-\alpha_1^2\alpha_2\mathrm{N}(\mathrm{N}+\mu_3)(\mathrm{N}\mu_3+1)(\mu_1-1)^2 \\
&\qquad+\alpha_0\alpha_2\alpha_3\mathrm{N}(\mathrm{N}+\mu_3)(\mathrm{N}\mu_1+1)(\mu_3-1)(\mu_1-1) \\
&\qquad-\alpha_0\alpha_2^2\mathrm{N}(\mathrm{N}+\mu_3)(\mathrm{N}\mu_1+1)(\mu_3-1)(\mu_1-1) \\
&\qquad-\alpha_0\alpha_1\alpha_3\mathrm{N}(\mathrm{N}+\mu_1)(\mathrm{N}\mu_1+1)(\mu_3-1)^2 \\
&\qquad+\alpha_0\alpha_1\alpha_2(\mathrm{N}\mu_1+1)(\mathrm{N}+\mu_1)(\mathrm{N}\mu_3+1)(\mathrm{N}+\mu_3).
\end{aligned}
$$

If we write

$$
\begin{aligned}
&\xi_1=(\mathrm{N}\mu_1+1)(\mathrm{N}+\mu_1), && \xi_3=(\mathrm{N}\mu_3+1)(\mathrm{N}+\mu_3), \\
&\eta_1=\mathrm{N}(\mu_1-1)^2, && \eta_3=\mathrm{N}(\mu_3-1)^2, \\
&\mathrm{P}=\mathrm{N}(\mu_3-1)(\mu_1-1)(\mathrm{N}+\mu_3)(\mathrm{N}\mu_1+1), &&
\end{aligned}
$$

the equation reduces to

$$
\begin{aligned}
&\psi_0\alpha_0\alpha_1\alpha_2\mu_1\mu_3(\mathrm{N}+1)^4\Psi \\
&=\alpha_1(\xi_3\alpha_2-\eta_3\alpha_3)(\xi_1\alpha_0-\eta_1\alpha_1)-\mathrm{P}\alpha_2(\alpha_0-\alpha_1)(\alpha_2-\alpha_3),
\end{aligned}
$$

which, if we proceed to put $\mu_1=\mu_3$, further reduces to the expression previously obtained.

Differentiating,

$$\left.\begin{aligned}&\frac{D^4 d\Psi}{\psi_0\mu_1\mu_3(N+1)^4}\\ &\quad=\{P\alpha_2\alpha_1(\alpha_2-\alpha_3)-\eta\alpha_1^2(\xi_3\alpha_2-\eta_3\alpha_3)\}\alpha_1\alpha_2 d\alpha_0\\ &\qquad+\{-P\alpha_2\alpha_0(\alpha_2-\alpha_3)+\eta_1\alpha_1^2(\xi_3\alpha_2-\eta_3\alpha_3)\}\alpha_0\alpha_2 d\alpha_1\\ &\qquad+\{P\alpha_2^2(\alpha_0-\alpha_1)-\eta_3\alpha_1\alpha_3(\xi_1\alpha_0-\eta_1\alpha_1)\}\alpha_0\alpha_1 d\alpha_2\\ &\qquad+\{-P\alpha_2(\alpha_0-\alpha_1)+\eta\alpha_1(\xi_1\alpha_0-\eta_1\alpha_1)\}\alpha_0\alpha_1\alpha_2 d\alpha_3.\end{aligned}\right\}\quad(32)$$

If the inner shell and the external radius are given, the best value of α_2^2 is deduced from the equation

$$\alpha_2^2=\frac{\eta_3}{P}\cdot\frac{\alpha_1\alpha_3}{\alpha_0-\alpha_1}(\xi_1\alpha_0-\eta_1\alpha_1).$$

Comparing this with equation (6) we see that the new value of α_2 is greater than that obtained when the permeability of the external shell is the same as that of the inner one if

$$\eta_3/P>1/\xi_1\,;$$

i. e., if $(\mu_3-1)(N+\mu_1)>(\mu_1-1)(N+\mu_3)$,

i. e., if $\mu_3>\mu_1$.

In like manner, if the outer shell and the internal radius of the inner shell are given, the best arrangement requires that

$$\alpha_1^2=\frac{P\alpha_2\alpha_0(\alpha_2-\alpha_3)}{\eta_1(\xi_3\alpha_2-\eta_3\alpha_3)},$$

which is greater than the corresponding value when the permeabilities of the shells are the same as that of the inner shell if

$$\frac{P}{\eta_1(\xi_3\alpha_2-\eta_3\alpha_3)}>\frac{\xi_1}{\xi_1\alpha_2-\eta_1\alpha_3},$$

i. e., if $\alpha_2\xi_1(P-\eta_1\xi_3)\quad\alpha_3\eta_1(P-\xi_1\eta_3)$.

But

$$P-\eta_1\xi_3=N(\mu_1-1)(N+\mu_3)(N+1)(\mu_3-\mu_1),$$
$$P-\xi_1\eta_3=N(\mu_3-1)(N\mu_1+1)(N+1)(\mu_1-\mu_3)\,;$$

and therefore the two sides of the inequality are always of opposite signs.

Hence, if the permeability of the outer shell is increased, the values of α_2 and α_1 corresponding to the best arrangement under the circumstances supposed increase, *i. e.,* the outer shell increases and the inner shell diminishes in thickness.

In the case, therefore, of material such as iron, if the field within the outer shell is such that the permeability of the outer shell is the greater (which will generally be the case when the enclosed space is protected), it will be better to make the external shell relatively thicker than the above calculation would indicate. Conversely, if the inner shell has the higher permeability its thickness should be increased.

There is no difficulty in eliminating α_1 and α_2 in turn from the relations obtained by equating the coefficients of $d\alpha_1$ and $d\alpha_2$ in (32) to zero, and thus finding the best values in any given case for which the equations can be solved by trial.

Summary.

In conclusion it is perhaps desirable to summarize the principal results of the foregoing discussion. In doing so I shall confine myself exclusively to the corresponding cases of a small magnet placed in the centre of the shells, and of shells placed in a uniform field, and thus the interpretation of the symbols is valid for these cases only.

The results depend upon two quantities, viz., the ratio of the outermost to the innermost radius of the series of shells. In the paper the cube of this quantity is indicated by the symbol $L=\alpha_0/\alpha_n=a_n^3/a_0^3$, where n refers to the outermost surface.

The other quantity ϵ is a function of the permeability. In the cases which I am now considering,

$$\epsilon=\frac{9\mu}{(2\mu+1)(\mu+2)}=\frac{9}{2\mu}\text{ approximately,}$$

when μ is large.

If we suppose the innermost radius of the shells (a_0) to be fixed, and the thickness of the space utilized for shielding to increase gradually from zero to a thickness t, the material employed is most efficient if fashioned in the form of a single shell until

$$\frac{t}{a_0}=(1+\epsilon)^{\frac{1}{3}}-1=\frac{3}{2\mu}\text{ approximately,}$$

if μ is large.

After that the material will be better employed if we divide it into separated shells; but the result obtained by the best use of the given quantity of material can be still further improved by filling up the space between the shells with more of the material, provided the thickness of the whole shell occupied by the shielding machinery is less than the value given by

$$\frac{t}{a_0}=(1+\sqrt{\epsilon})^{\frac{1}{3}}-1=\sqrt{\frac{2}{\mu}},$$

if μ is large.

After this limit is passed it becomes positively injurious to fill the whole of the shielding space with the permeable material.

Better results can always be obtained by two shells with the same internal and external radii.

If the ratio of these radii ($a_3/a_0=\sqrt[3]{L}$) is given, the best arrangement of two shells is defined by the equations

$$\frac{a_3}{a_2}=\frac{a_1}{a_0}=\lambda^{\frac{1}{3}},$$

where λ is given by the equation

$$\lambda^2(\lambda-1)-L(\lambda-1+\epsilon)=0.$$

The best arrangement for three shells, if the ratio of the innermost and outermost radii is given ($a_5/a_0=\sqrt[3]{L}$), is defined by the equations:

$$\frac{a_5}{a_4}=\frac{a_1}{a_0}=\lambda^{\frac{1}{3}},$$

$$\frac{a_4}{a_3}=\frac{a_3}{a_2}=\frac{a_2}{a_1}=L^{\frac{1}{9}}\lambda^{-\frac{2}{9}},$$

where λ is defined by the equation

$$\lambda^{\frac{5}{3}}(\lambda-1)-L^{\frac{1}{3}}(\lambda-1+\epsilon)=0.$$

In both cases, when the permeability is great the best arrangement is that in which the radii of the successive bounding surfaces of the shells are in Geometrical Progression.

The ratio of the shielded to the unshielded field when the best arrangement is adopted is

$$\frac{L^{\frac{2}{3}}\epsilon^2}{(\lambda-1)^2\{\lambda^4-L(1-\epsilon)\}} \quad \text{and} \quad \frac{L^{\frac{4}{5}}\epsilon^3}{(\lambda-1)^4\{\lambda^{\frac{6}{5}}-L^{\frac{1}{5}}(1-\epsilon)\}}$$

for two and three shells respectively, λ being in each case determined from the equation given above as appropriate to that case.

When the permeability is great and the shells are not very thin, these expressions become

$$\frac{L\epsilon^2}{(\sqrt[3]{L}-1)^3} \quad \text{and} \quad \frac{L\epsilon^3}{(\sqrt[5]{L}-1)^5}, \text{ respectively.}$$

The conditions for the best arrangement in the form of two shells, when (1) the innermost radius and the total volume of the permeable material, and (2) the innermost and outermost radii and the total volume of permeable material are given, are less symmetrical but are set forth above.

In any case of an arrangement in the form of two shells of materials of different permeability, in order to obtain the best result, the thickness of the shell formed of the more permeable material should be greater, and that of the shell of less permeable material should be less than is given by the above rules.

The equations are obtained by which the best arrangement can be calculated, if the permeabilities and external and internal radii are given.

DISCUSSION.

Prof. Minchin said the mathematical results were very simply expressed. Although the work was apparently restricted to zonal spherical harmonics, some of the important formulæ apply equally to general spherical harmonics. Referring to the difficulty of shielding by single thick shells, he pointed out that the equation giving the relation between the shielded and the unshielded fields with different thicknesses of shell represented a hyperbola with its asymptotes parallel to the axes; hence the shielding tended to a definite limit as the thickness increased indefinitely.

Mr. Evershed said he had been engaged for the last two years on the subject of magnetic shielding with a view to screening measuring instruments from external fields. In such cases it was not possible to use closed shells, and this introduced trouble. The best result he had yet obtained was to reduce the disturbance to about one-fifth. Another difficulty was introduced by the fact of the shield being magnetized by the current passing through the coil, and owing to hysteresis, the permeability was different according as the magnetization increased or decreased. By using an outer iron shell a great improvement had been effected. To obtain the best results it was important to have no joints in the shields. A coil frame with two shields of bent iron was exhibited.

Mr. J. Swinburne remarked that the subject divided itself into two: shielding instruments and shielding sources. If a dynamo itself be shielded, this did not prevent the currents in the leads producing magnetic disturbances. This was very important in ships. By using an alternator with revolving fields all disturbances could be avoided.

Dr. C. V. Burton enquired whether, by considering the hydrodynamical analogue of a porous material, the case of perforated shells could be elucidated.

Mr. A. P. Trotter wished to know if the homogeneity of the shield was of much consequence. At Oxford it had been found that a screen of four inches of scrap iron was better than boiler-plate.

Mr. Blakesley asked if the effect of moving a magnet sideways in a sphere had been observed. He thought the mathematics developed in the paper would be useful in working out the magnetic theory of the earth.

Prof. S. P. Thompson thought that taking the permeability as constant would not be quite correct, for μ was a function of the magnetization. Hence in the cases considered the outer shell would be the more permeable.

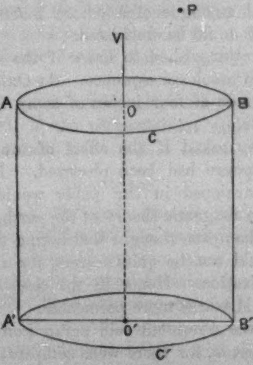
In his reply the President said scrap iron in contact was not like clear space, for there were comparatively free paths for the induction at the points of contact. As regards the shielding of the dynamo at Greenwich, Mr. Christie had written to say that the credit was due to the makers of the machine and shields, Messrs. Johnson and Phillips.

XXXIV. *Calculation of the Magnetic Field of a Current running in a Cylindrical Coil.* *By* Professor G. M. Minchin, *M.A.*[*]

Let there be a series of very close circular currents running in the same sense and lying on a right cylinder of radius a whose axis is OO′ (fig. 1), and let it be required to find the magnetic potential of this system at any point, P, in space.

Replace each of these circular currents by its equivalent magnetic shell, which we shall take as a uniform circular plate coinciding with the aperture of the circle. Supposing the currents to circulate in the sense ACB, the upper surface of each plate (as seen in the figure) will be positive and the lower negative. Each circle being touched all round by the one below it, the negative surface of any plate will coincide with the positive of the one next below it; so that we shall be left with a terminal positive plate, ACB, and a terminal negative plate, A′C′B′, as the approximate equivalent of the

Fig. 1.



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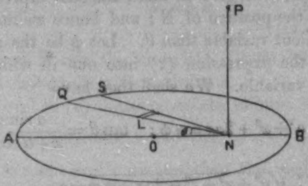
[*] Read December 8, 1893.

given coil of wire. We shall now calculate the potential produced at any point, P, by a uniform thin plate represented by AQB (fig. 2), the surface-density of attracting matter on this plate being m.

Fig. 2.



Of course it is well known that this potential can be exhibited by a series of Zonal Harmonics proceeding by powers, positive or negative, of the ratio $\frac{OP}{a}$, where O is the centre of the plate and a its radius, according as P is near O or distant from it; but when OP is not *very* much greater or *very* much less than a, the series is inconvenient, owing to the enormous number of terms that have to be taken to give a good approximation.

Hence we shall not use Spherical Harmonics.

Let PN be the perpendicular on the plate from P; let AB be the diameter in which it is cut by the plane through the axis of the plate and P; let Q, S be two very close points on the circumference, NQ making the angle θ with NA. Then we shall suppose the plate broken up into triangular strips such as QNS, and calculate the potential of each strip at P.

Let L be any point on NQ, let $NL=\xi$; then the potential of the element $m\xi d\xi d\theta$ at P is $\frac{m\xi d\xi d\theta}{LP}$, or $md\theta\frac{\xi d\xi}{\sqrt{z^2+\xi^2}}$, where $PN=z$. If $NQ=r$ we find by integration that the potential produced by the strip QNS is

$$md\theta(\sqrt{r^2+z^2}-z), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the potential, V, of the whole plate is

$$2m\int(\sqrt{r^2+z^2}-z)d\theta. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now if, as in the figure, the point N falls within the plate, the limits of θ are 0 and π; if N falls on the edge of the plate, at B, the limits are 0 and $\frac{\pi}{2}$; and if it falls outside the plate, the limits are 0 and 0. Taking θ as the independent variable would, then, give us three different expressions for V, according to the position of N; and hence we must choose a more convenient variable than θ. Let ϕ be the angle QOA, and change the expression (2) into one in which ϕ is the independent variable. We shall then have

$$r^2 = a^2 + x^2 + 2ax\cos\phi\,; \quad \tan\theta = \frac{a\sin\phi}{x + a\cos\phi}\,;$$

so that

$$\mathrm{V} = 2ma\int_0^\pi \frac{\sqrt{a^2+x^2+z^2+2ax\cos\phi} - z}{a^2+x^2+2ax\cos\phi}(a + x\cos\phi)d\phi\,;$$

or if we put $\mathrm{D} = a^2 + x^2 + 2ax\cos\phi$,

$$\frac{\mathrm{V}}{m} = \int_0^\pi \frac{\sqrt{z^2+\mathrm{D}} - z}{\mathrm{D}}(\mathrm{D} + a^2 - x^2)d\phi$$

$$= \int_0^\pi \sqrt{z^2+\mathrm{D}}\,d\phi + (a^2-x^2)\int_0^\pi \frac{\sqrt{z^2+\mathrm{D}}}{\mathrm{D}}d\phi - z\int_0^\pi \left(1 + \frac{a^2-x^2}{\mathrm{D}}\right)d\phi.$$

Now

$$(a^2-x^2)\int_0^\pi \frac{d\phi}{\mathrm{D}} = \pi\,;$$

so that

$$\frac{\mathrm{V}}{m} = -2\pi z + \int_0^\pi \sqrt{z^2+\mathrm{D}}\,d\phi + (a^2-x^2)\int_0^\pi \frac{\sqrt{z^2+\mathrm{D}}}{\mathrm{D}}d\phi$$

$$= -2\pi z + \int_0^\pi \sqrt{z^2+\mathrm{D}}\,d\phi + (a^2-x^2)\int_0^\pi \frac{d\phi}{\sqrt{z^2+\mathrm{D}}}$$

$$+ (a^2-x^2)z^2\int_0^\pi \frac{d\phi}{\mathrm{D}\sqrt{z^2+\mathrm{D}}}. \quad (3)$$

Again, if we put $\phi = 2\omega$, we have

$$\mathrm{D} = (a+x)^2 - 4ax\sin^2\omega\,;$$

and if the distances PA, PB are denoted by ρ, ρ', respectively, we have

$$4ax = \rho^2 - \rho'^2,$$

so that

$$z^2 + \mathrm{D} = \rho^2 - (\rho^2 - \rho'^2)\sin^2\omega.$$

Let

$$k^2=1-\frac{\rho'^2}{\rho^2}\,;\quad k'^2=\frac{\rho'^2}{\rho^2}\,;\quad \Delta\omega=\sqrt{1-k^2\sin^2\omega}. \quad . \ (4)$$

Then (3) becomes

$$\frac{V}{m}=-2\pi z+2\rho E+\frac{2}{\rho}(a^2-x^2)K+2\frac{z^2}{\rho}\,\frac{a-x}{a+x}\cdot\int_0^{\frac{\pi}{2}}\frac{1}{1-\frac{4ax}{(a+x)^2}\sin^2\omega}\cdot\frac{d\omega}{\Delta\omega}, \quad (5)$$

where E and K are the complete elliptic integrals of the second and first kinds with modulus k.

The integral in (5) is the complete elliptic integral of the third kind with modulus k and parameter $-\frac{4ax}{(a+x)^2}$. This parameter is numerically greater than the modulus; and we shall find it convenient to convert the integral into one in which the parameter is less than the modulus by the well-known rule that a function with parameter n can be converted into one with parameter $\frac{k^2}{n}$. If the angles PBA and PAB are denoted by θ and θ', respectively, we see that n, the parameter in (5), is $-\frac{k^2}{\cos^2\theta'}$; so that the new parameter will be simply $-\cos^2\theta'$.

Now we have the general result that

$$\Pi(n,\epsilon)+\Pi\left(\frac{k^2}{n},\epsilon\right)=\frac{1}{\sqrt{\alpha}}\tan^{-1}\left(\frac{\sqrt{\alpha}\tan\epsilon}{\Delta\epsilon}\right)+K(\epsilon),$$

where ϵ is the amplitude of each of the two functions of the third kind (denoted by Π), and $\alpha=(1+n)\left(1+\frac{k^2}{n}\right)$. Hence for complete functions $\left(\epsilon=\frac{\pi}{2}\right)$ we have

$$\Pi(n)+\Pi\left(\frac{k^2}{n}\right)=\frac{\pi}{2\sqrt{\alpha}}+K.$$

But here

$$\alpha=\left(1-\frac{k^2}{\cos^2\theta'}\right)\sin^2\theta'=\left(\frac{a-x}{a+x}\right)^2\cdot\frac{z^2}{\rho^2}\,;$$

so that

$$\Pi\left(-\frac{k^2}{\cos^2\theta'}\right)=\frac{\pi}{2}\,\frac{\rho}{z}\,\frac{a+x}{a-x}+K-\Pi(-\cos^2\theta'), \quad . \ . \ (6)$$

and (5) becomes

$$\frac{V}{m}=-\pi z+2\rho E+2\rho\frac{a-x}{a+x}\,.\,K-2\frac{z^2}{\rho}\frac{a-x}{a+x}\Pi(-\cos^2\theta', k). \quad (7)$$

This expression holds without ambiguity for all positions of the point P, and it shows that for all points

on the axis, OV, of the plate, . $\frac{V}{m}=2\pi(\rho-z)$, . . (8)

on the perpendicular through B, $\frac{V}{m}=2\rho E-\pi z$, . . (9)

in the plane, between O and B, . $\frac{V}{m}=2\rho E+2\rho' K$, . (10)

in the plane, beyond B, ... $\frac{V}{m}=2\rho E-2\rho' K$; . (11)

so that the points, occupying any of these positions, at which V has any assigned value can be easily found. Thus, to find the point on OV at which $\frac{V}{m}$ has the value C, we have for this point

$$\rho=z+\frac{C}{2\pi}.$$

Hence draw below AB, parallel to it and at the distance $\frac{C}{2\pi}$, a right line, meeting VO produced in O′; then the perpendicular to AO′ at its middle point meets OV in the required point.

Now every complete elliptic integral of the third kind can be expressed in terms of complete and incomplete functions of the first and second kinds. Thus, for a complete function with the parameter $-m$ it is known that, if we put $m=-k'^2\sin^2\epsilon$,

$$\Pi(-m, k)=K+\frac{\Delta'_\epsilon}{k'^2\sin\epsilon\cos\epsilon}\left\{\frac{\pi}{2}+KK'_\epsilon-KE'_\epsilon-EK'\right\}, \quad (12)$$

where Δ'_ϵ stands for $\sqrt{1-k'^2\sin^2\epsilon}$, and K'_ϵ, E'_ϵ stand for incomplete functions of the first and second kinds with modulus k' and amplitude ϵ.

In the present case,

$$m = \cos^2\theta', \quad k' = \frac{\rho'}{\rho} = \frac{\sin\theta'}{\sin\theta};$$

$\therefore\ \epsilon = \theta$, and

$$\Pi(-\cos^2\theta', k) = \mathrm{K} + \frac{\cos\theta'}{k'^2\sin\theta\cos\theta}\left\{\frac{\pi}{2} + \mathrm{KK}'_\theta - \mathrm{KE}'_\theta - \mathrm{EK}'_\theta\right\}; \quad (13)$$

so that (7) becomes

$$\frac{\mathrm{V}}{2m} = z\{\mathrm{KE}'_\theta + \mathrm{EK}'_\theta - \mathrm{KK}'_\theta - \pi\} + \rho\mathrm{E} + \rho'\cos\theta\cos\theta'.\mathrm{K}. \quad (14)$$

It is evident that we may define the position of any point, P, in the plane of the figure by means of the two coordinates k and θ. Thus we have

$$\rho = \frac{2a}{\Delta'_\theta + k'\cos\theta},$$

$$\cos\theta' = \Delta'_\theta,$$

$$z = k'\rho\sin\theta, \quad \rho' = k'\rho.$$

Hence

$$\frac{\mathrm{V}}{4m} = \frac{a}{\Delta'_\theta + k'\cos\theta}\{\mathrm{E} + k'\sin\theta\,(\mathrm{KE}'_\theta + \mathrm{EK}'_\theta - \mathrm{KK}'_\theta - \pi) + k'\cos\theta\,.\,\mathrm{K}\Delta'_\theta\}. \quad (15)$$

This, then, is the expression for the potential at any point in terms of the coordinates (k, θ) of the point. In particular, it gives the value (9) for any point on the perpendicular through B to the plate, since for such a point $\theta = \frac{\pi}{2}$, and then the coefficient of $k'\sin\theta$ within the brackets is equal to $-\frac{\pi}{2}$, by Legendre's well-known relation between the complete complementary functions, viz.,

$$\mathrm{KE}' + \mathrm{EK}' - \mathrm{KK}' = \frac{\pi}{2},$$

whatever the modulus k may be.

From (14) we can derive an expression for the conical angle subtended at any point, P, in space by a circle, *i. e.*, for the magnetic potential due to a current coinciding with the circle. It is well known that this conical angle is numerically equal to the component of the attraction, perpendicular to the plate, at P due to a uniform circular plate coinciding with the aperture of the circle—a result which is evident from the principle that the current can be replaced by a magnetic shell, or thin plate, the upper and lower surfaces of which are, of course, of opposite signs. But the resultant potential of these two indefinitely close plates is the difference between the value of V in (14) and the value which (14) assumes when $z+\Delta z$ is substituted for z; that is, the magnetic potential at P due to the current is $-\frac{dV}{dz}\,.\,\Delta z$, and the strength of the magnetic shell is $m\,.\,\Delta z$, which is i, the current in the circle; so that the magnetic potential is i multiplied by minus the differential coefficient of the right-hand side of (14) with respect to z.

Denote the function $\pi+KK'-KE'_\theta-EK'_\theta$ by the symbol Λ_θ, and for simplicity in the differentiation with respect to z (x being constant) write (14) in the form

$$\frac{V}{2m}=-z\Lambda_\theta+\rho E+(a^2-x^2)\frac{K}{\rho}. \quad . \quad . \quad . \quad (16)$$

Now

$$\left.\begin{aligned}\frac{dk}{dz}&=-\frac{kk'}{\rho}\sin\theta; \quad \frac{dk'}{dz}=\frac{k^2}{\rho}\sin\theta;\\ \frac{d\rho}{dz}&=k\sin\theta; \quad \frac{d\theta}{dz}=\frac{1}{k'\rho}\cos\theta;\end{aligned}\right\} \quad . \quad . \quad (17)$$

and, regarding P as determined by the coordinates (z, x) instead of (k, θ), we have

$$\frac{d}{dz}=\frac{dk}{dz}\cdot\frac{d}{dk}+\frac{d\theta}{dz}\cdot\frac{d}{d\theta};$$

but

$$\frac{d\Lambda}{dk}=\frac{K-E}{k}\,\frac{\sin\theta\cos\theta}{\Delta'_\theta},$$

$$\frac{d\Lambda}{d\theta}=\frac{Kk'^2\sin^2\theta-E}{\Delta'_\theta};$$

therefore

$$\frac{d\Lambda}{dz}=-\frac{1}{k'\rho}E\Delta'_\theta\cos\theta,$$

and we find $\frac{d}{dz}$ of the right-hand side of (16) equal to $-\Lambda_\theta+k'K\sin\theta$; so that if Ω is the conical angle subtended at P by the circle, or the magnetic potential per unit current in the circle, we have the very simple expression

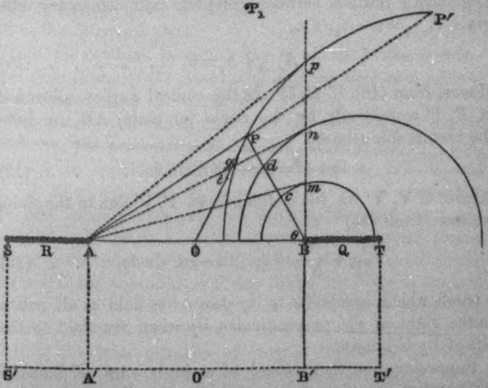
$$\Omega=2\Lambda_0-2k'\mathrm{K}\sin\theta. \quad . \quad . \quad . \quad . \quad (18)$$

Again, supposing that the depth, OO′ (fig. 1), of a coil consisting of a single series of circular currents is small compared with the distance of the point P from any part of it, the two terminal plates, ACB, A′C′B′ may be considered as close together, and the potential of the coil at P is the value of V in (16) minus the value obtained by putting $z+h$ for z, where $h=$OO′. Hence the potential in such a case is $-mh\Omega$, i. e., *at any point in space whose distance from every part of a coil is great compared with the depth of the coil, the potential is*

$$2mh(\Lambda_0-k'\mathrm{K}\sin\theta). \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The modulus k which appears in these equations, being $\left(1-\frac{\rho'^2}{\rho^2}\right)^{\frac{1}{2}}$, is constant at all points for which $\frac{\rho'}{\rho}$ is constant; *i. e.*, at all points on any circle which cuts that described on AB as diameter orthogonally. The circles which cut this latter orthogonally, having their centres on AB, are most readily drawn by joining A to points, $m, n, p, \ldots$ (fig. 3) on the perpendicular at B to AB, and drawing perpendiculars at

Fig. 3.



$m, n, p, \ldots$ to Am, An, Ap, ...; the points of intersection of AB produced with these perpendiculars are the centres of the circles. If C is the centre of the orthogonal circle through m, we know that the constant, $1-\frac{\rho'^2}{\rho^2}$, on this circle is $\frac{AB}{AC}$, *i. e.*, $\cos^2 mAB$. Hence if $mAB=\beta$, we have

$$k=\cos\beta; \qquad k'=\sin\beta. \quad . \quad . \quad . \quad . \quad (20)$$

The field due to the plate AB is most readily mapped out by describing a large number of very close circles of the orthogonal system for a regular gradation of the values mAB, nAB, pAB, ... of β, drawing a line BP in the assigned direction θ, and from Legendre's tables of Elliptic Integrals taking out the values of K, E, K'_θ, E_θ.

The properties of the orthogonal circles lead to some simple results with regard to potentials. Thus, if any line, AP, is drawn from A cutting any circle of the series in P and P′, the lines joining P and P′ to B are equally inclined to AB, *i. e.*, $\angle ABP'=\pi-\theta$.

Now if in Λ_θ we put $\pi-\theta$ for θ, we have, in virtue of Legendre's relation between complete complementary integrals, $\Lambda_{\pi-\theta}=\pi-\Lambda_\theta$, *i. e.*,

$$\Lambda_{\pi-\theta}+\Lambda_\theta=\pi. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Hence, from (18), if Ω, Ω' are the conical angles subtended at P, P′ respectively by the circle (or plate) AB, we have the remarkable relation

$$\Omega+\Omega'=2\pi-4k'K\sin\theta. \quad . \quad . \quad . \quad . \quad (22)$$

Again, if V, V′ are the potentials at P, P′ due to the plate, we have from (15)

$$\frac{V}{AP}+\frac{V'}{AP'}=2m(2E-\pi k'\sin\theta), \quad . \quad . \quad . \quad (23)$$

a result which enables us to lay down the field at all points to the right of the perpendicular Bp when the field to the left of Bp is known.

Supposing now that instead of a single wire of diameter AB, we have a series of wires forming a coil contained between the diameter AB and the diameter ST, *i. e.*, the breadth of the coil is BT or AS; then in calculating the potential at

P we shall have to find the potentials due to a series of circular plates, each of surface-density *m*, and to add these potentials together. But observe that the potential at P due to any plate, AB, of radius a is of the form

$$a \, . \, \phi(k, \theta), \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where $\phi(k, \theta)$ is the coefficient of a in (15), and $\phi(k, \theta)$ is a function of θ and θ', the angles PAB and PBA. Hence if we take a plate of radius OQ, and from B draw Bq parallel to QP and meeting OP in q, the potential of this plate at P is to the potential of the plate AB at q as OQ is to OA; for, if AR=BQ, the angles qBA and qAB are equal, respectively, to PQR and PRQ. Hence, if r=OQ and V_q is the potential at q due to the plate AB (of radius a), the resultant potential at P due to the series of plates of radii extending from OB to OT is

$$\frac{1}{a}\Sigma r \, . \, V_q, \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

the points q on OP ranging from t to P, where Bt is parallel to PT.

Of course any plate of the series may be taken instead of AB as the reference plate.

Thus, *the resultant potential, due to all the plates, is calculated from values of the potential of any one plate at a series of points ranged along the radius vector* OP.

Pass now to the consideration of the practical problem in hand, viz., the potential at P due to a coil of depth OO′, *i. e.*, we have to consider the whole of the spaces BTT′B′ and ASS′A′ filled with wire traversed by a current of strength *i*. We have already seen that we have to subtract from the potential at P due to a series of uniform attracting plates, each of surface-density *m*, ranging from the radius OB to the radius OT, the potential at P due to the lower series, each of surface-density *m*, and ranging from radius OB′ to radius OT′. It merely remains to express *m* in terms of current-density. If C is the total quantity of current traversing (at right angles to the plane of the paper) a unit area (square centimetre) of the space BTT′B′, the quantity flowing in a filament of depth dy and breadth dr is $Cdydr$. Now this filament is replaced by the magnetic shell of radius r and thickness dy;

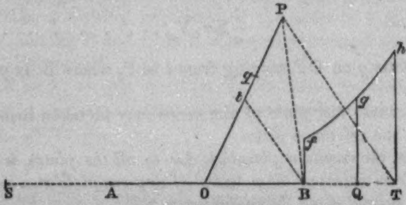
and since we know that the strength of the shell is equal to the current in the filament, we have $m \, . \, dy = C dy dr$, $\therefore \, m = C dr$; hence (25) becomes, from (15),

$$4C\int r \, . \, \phi(k, \theta) dr, \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

which is the potential due to the upper series of plates, OB, . . . OT.

This quantity may be graphically represented and calculated as follows. Let a very close series of curves representing a series of constant values of the function $\phi(k, \theta)$ for the plate AB be drawn; draw OP, and at each point, T, Q, . . . B, of the breadth BT of the coil draw an ordinate, Th, Qg, . . . Bf (fig. 4), equal to the product of r and the value of $\phi(k, \theta)$ at the corresponding point, t, q, . . . P, of the line OP: these ordinates will form by their extremities a curve, $hg \ldots f$, the area of which multiplied by four times the current-density in the space occupied by the coil is the potential at P due to the upper series of plates, OB, OQ, . . . OT.

Fig. 4.



Now if we take the point P_1 such that PP_1 is equal and parallel to OO′, the depth of the coil, the potential at P due to the lower series of plates, OB′, . . . OT′, is equal to that at P_1 due to the upper series. Hence, if A and A_1 are the areas of the curve $hg \ldots f$ and the corresponding curve for the point P_1, the total potential at P due to the complete coil is

$$4C(A - A_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The curve *hgf* passes, of course, through the point O; and when the line OP coincides with the axis, OO′, of the coil, the curve is an hyperbola; for, in this case

$$\phi(k,\theta)=\frac{\pi}{2}\cdot\frac{1-\sin\theta}{\cos\theta},$$

and $r=\mathrm{OP}\cot\theta$.

We may, if we please, express r in terms of (k,θ), and draw the curve *hgf* by a different rule. Thus,

$$r=\mathrm{OP}\,.\,\frac{\Delta'_\theta+k'\cos\theta}{\sqrt{(\Delta'_\theta-k'\cos\theta)^2+4k'^2\sin^2\theta}},$$

and we can make the ordinate of the curve equal to

$$\mathrm{OP}\,.\,\frac{\mathrm{E}-k'\sin\theta\,.\,\Lambda_\theta+k'\mathrm{K}\cos\theta\,.\,\Delta'_\theta}{\sqrt{(\Delta'_\theta-k'\cos\theta)^2+4k'^2\sin^2\theta}}.$$

DISCUSSION.

Prof. Perry hoped the work Prof. Minchin had done so well for circles and cylinders would be extended to cylindrical coils of rectangular cross-section. It was most important to be able to find the shape of the field produced by such coils.

Prof. S. P. Thompson enquired if there was any way of deducing the expression for the magnetic force at a point other than that given in the paper on the "Magnetic Field of a Circular Current" (*antea*, p. 204).

In reply, Prof. Minchin explained how the formula followed at once from the fundamental theorem that magnetic force is the cure of the vector potential. This was based on Laplace's expression for the force between a magnetic pole and an element of current, which had been proved experimentally.

XXXV. *A Potentiometer for Alternating Currents.*

By James Swinburne*.

Argument.

A direct pressure is balanced against an alternating pressure by means of a differential electrometer with single-fibre suspension, a null method being employed. An alternating and a direct current are similarly compared by means of a differential dynamometer with the controlling spring removed.

One of the chief difficulties in the way of the accurate measurement of alternating pressures and currents arises in the calibration of the instruments. Some of the alternating-current instruments, such as those on the electro-dynamometer principle, give the same reading whether a direct or alternating current is used; and electrometers and some forms of electromagnetic instruments with soft iron cores are also adapted equally well for direct and alternating-current work. This is not the case, however, with a large class of instruments, and the only method at present available for calibrating such instruments is comparing them with another instrument which has been calibrated by means of a direct current. Such a practice allows errors to creep in. No careful electrician would trust to direct-current instruments calibrated second-hand, and accuracy is just as important in alternating-current work. In addition to this, the collection of cells and resistances which are now included under the name "potentiometer" is exceedingly useful in any laboratory where it can be permanently set up, as it measures currents and pressures throughout an enormous range with an accuracy that is quite unattainable by means of any other forms of voltmeter or ampere-meter. It is therefore of the highest scientific importance to be able to extend the potentiometer method into alternating-current work, so that comparisons can be made directly with a standard cell.

In October 1891† I described two forms of alternating-current so-called ohmmeter. These instruments measured either a quantity R such that E^2/R was the power when E was the effective or virtual pressure, or a quantity r such that Cr^2 was the power when C was the virtual or effective current. Of course in a circuit with capacity or self-induction, R and r are not equal. One form of alternating ohmmeter was electrostatic, the other electromagnetic. The electrostatic ohmmeter can be coupled up in such a way as to compare two pressures, and one of these may be alternating and the other direct. The electrostatic instrument has the advantage over the electromagnetic in requiring only one connexion to the moving system, or needle, and that connexion has not to carry any appreciable current. The needle may therefore be suspended by means of a single silk fibre, connexion being made through a hanger dipping into water, and this hanger may end in a vane to make the instrument dead-beat. As the ohmmeter does not need any torsional control, but should be, on the contrary, as free as possible, this form of instrument can be made exceedingly sensitive. For potentiometer work it is best not to use the instrument for giving the ratio of two electromotive forces, but to design it as a differential galvanometer which shows whether the pressures are equal or not.

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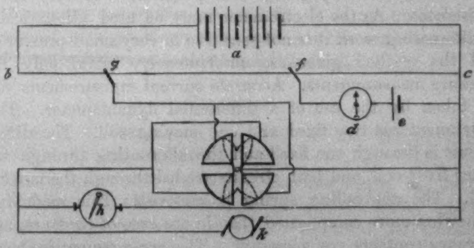
* Read December 8, 1893.

† 'Industries,' October 30, 1891.

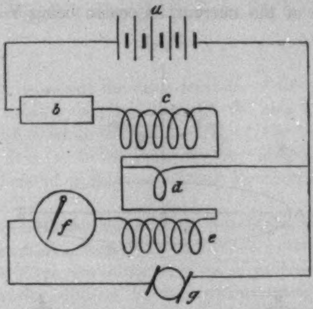
Fig. 1.



The disposition of the whole apparatus is shown diagrammatically in fig. 1. In this scheme *a* is a battery maintaining a current in the resistance *b c*, which is regulated by an adjustable resistance so that the standard cell *e* is balanced when its contact *f* is touched down at the mark corresponding to its electromotive force at the temperature at which it stands; *k* is an alternating dynamo, and *h* is the alternating voltmeter to be calibrated. The differential electrometer is connected up as shown. The double fishtail-shaped needle is pulled one way by a force varying as the square of the direct electromotive force, and the other way by a force varying as the square of the alternating pressure. The fishtail shape is necessary to ensure that the needle is in stable equilibrium when the forces are equal. Fig. 1 is, of course, a mere diagram. Resistance-coils for adjustment are left out, and *b c* is shown as a stretched wire, whereas it is made up of resistance-coils. As to sensitiveness, it depends on the construction of the electrometer; but, as already explained, there is no controlling force except that due to the suspending fibre. Take as an example an instrument that will indicate one volt when there is no pressure on the other side: it will admit of the comparison of two pressures of approximately 100 volts within one in ten thousand; and it will serve for comparing 2000 volts within one in four million. To find the true zero of the instrument the four quadrants are connected together. The key for making this connexion is not shown.

One of the great advantages of the potentiometer method in direct-current work is the ease with which large currents can be measured by the fall of potential over very small resistances. As the electrometer must be used idiostatically in alternating work it is not sensitive to very small pressures; and the method given is, unfortunately, useful only for pressure measurements. Accurate current measurements can be taken by the use of a differential dynamometer. This instrument has two fixed and one moving coil. The direct circuit is through one fixed and the alternating through the other fixed coil, and both circuits are led through the moving coil. The controlling spring is removed. This method is somewhat more complicated than in the case of pressure, and mercury contacts are necessary. The arrangement is shown in fig. 2. The battery, *a*, supplies a direct current, which is measured by the fall of potential over the low resistance, *b*, and which passes in one of the fixed coils, *c*, and the moving coil, *d*. The alternating-current circuit is led through the circuit, *f*, whose current is to be measured, and through the other fixed coil, *e*, and the moving coil, *d*. There is a slight error due to the whole of the alternating circuit being in shunt to the moving coil considered as part of the direct-current circuit, and *vice versâ*; but this is very minute. If *c* and *e* are wound close together so that there is no appreciable time-lag in any current induced in *c* by *e*, the small error due to the mutual induction of these coils cancels out.

Fig. 2.



Of course a double dynamometer might be used, but then four mercury connexions are needed. It is not unlikely that a method with no mercury contacts can be made available, and greater accuracy could then be attained.

DISCUSSION.

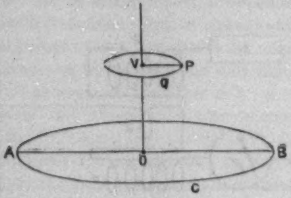
Prof. S. P. Thompson inquired if the fishtail-shaped needle of the electrometer was novel.

Mr. Blakesley said the author had mentioned the needle previously. He (Mr. Blakesley) thought the name "potentiometer" was not very suitable. In effect the so-called measurement of *pressures* was a comparison of two *powers*.

XXXVI. *Calculation of the Coefficient of Self-Induction of a Circular Current of given Aperture and Cross-Section.* By Professor G. M. MINCHIN, *M.A.**

LET ACB (fig. 1) represent a circular wire in which a current of strength i is circulating; let O be its centre and OV its axis (perpendicular to its plane); let P be any point in space and through P describe a circle, PQ, parallel to the plane of the current, its centre being V on the axis.

Fig. 1.



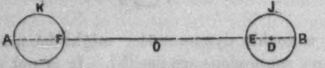


Fig. 2.

It is required to calculate the normal flux of magnetic force passing through the circle PQ. If $VP=x$, and the vector potential of the current at P is G (this latter being, of course, perpendicular to VP and parallel to the plane of the current), the component, Z, of the magnetic force at P parallel to OV is given by the expression

$$Z=\frac{dG}{dx}+\frac{G}{x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(See my paper on the "Magnetic Field of a Circular Current," *antea*, p. 204). This can be written

$$Z=\frac{1}{x}\,\frac{d(Gx)}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

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* Read December 8, 1893.

The function $G.x$ is the same as Stokes's *current function* which exists for fluid motion which is symmetrical about an axis. (See Basset's 'Hydrodynamics,' vol. i. p. 12.)

Taking a circular strip of radius x and breadth dx at P, the flux of force through the strip is $2\pi Zx\,dx$, i.e.,

$$2\pi\frac{d(Gx)}{dx}\,.\,dx.$$

Hence, integrating this from $x=0$ to $x=VP$, we find that the total normal flux through the circle PQ is the value of

$$2\pi Gx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

at P.

Let fig. 2 represent the cross-sections of the wire at A and B in fig. 1 made by a plane through the axis OV, the radius of each being c, while the radius, OD, of the central filament of the wire is a (as in my paper on the "Magnetic Field close to the Surface of a Wire conveying an Electrical Current," *antea*, p. 379).

Then we shall calculate *the total normal flux of force through any surface which is intersected once in the positive direction by every tube of force emanating from the given current.*

This quantity, divided by the current-strength, is the coefficient of self-induction of the current. Taking the general case, viz., that in which the current-density at every point in the cross-section of the wire varies inversely as the distance of this point from the axis OV—we have found (*antea*, p. 398) that at any point, P, close to the wire

$$Gx=4ai\left\{\frac{L}{2}-1-\frac{m}{4a}\cos\phi\left[L-1-\frac{c^2}{4m^2}\right]\right.$$
$$+\frac{1}{16a^2}\left[L\left(\frac{3c^2}{4}-\frac{m^2}{2}\right)+\frac{m^2}{2}-\frac{15c^2}{4}\right.$$
$$\left.\left.-\cos 2\phi\left\{L\frac{m^2}{2}+\frac{c^2}{4}-m^2-\frac{c^2}{8m^2}\right\}\right]\right\}\cdot \quad (4)$$

where $m=PD$, $\phi=\angle PDA$, $L=\log_e\dfrac{8a}{m}$.

The surface through which we shall take the flux of force is that which is represented in section by BJEOFKA, *i.e.*, a surface consisting of the upper half portion of the anchor-ring formed by the wire and of its central aperture (which latter is a circle whose diameter is FE). Obviously this surface is intersected by all the tubes of force. Any surface starting from B and going round to A, *i. e.*, any surface having the circle of diameter AB for bounding edge, would do equally well, so far as the above condition is concerned; but the calculation is simpler for the first.

The flux through the aperture FE is, then, the value of $2\pi G . x$ at E, which is obtained by putting $m=c$, $\phi=0$ in the above.

Thus the flux through the aperture is

$$8\pi a i\left\{\frac{L}{2}-1-\frac{c}{4a}\left(L-\frac{5}{4}\right)-\frac{c^2}{128a^2}(2L+19)\right\}. \quad . \ (5)$$

To calculate the normal flux through the upper half of the anchor-ring, we must take the value of the magnetic potential, Ω, at any point close to the ring. This is the resultant conical angle subtended by the circuit at the point, multiplied by i; and it is therefore (*antea*, p. 392) given by the equation

$$\Omega=i\left\{2(\pi-\phi)-\frac{\sin\phi}{a}\left(mL+\frac{c^2}{4m}\right)+\frac{1}{16a^2}\left[8c^2(\pi-\phi)\right.\right.$$
$$\left.\left.-\left\{(6L-5)m^2+c^2+\frac{c^4}{2m^2}\right\}\sin 2\phi\right]\right\}. \quad . \ (6)$$

Now the normal force at any point on the anchor-ring is the value of $-\frac{d\Omega}{dm}$ with $m=c$; it is therefore

$$i\left\{\left(L-\frac{5}{4}\right)\frac{\sin\phi}{a}+\frac{c}{16a^2}(12L-17)\sin 2\phi\right\}. \quad . \ (7)$$

If P is any point on the anchor-ring, the distance of P from OV is $a-c\cos\phi$, and the area of a narrow circular strip of the ring parallel to the plane of the aperture of the ring is $2\pi(a-c\cos\phi) . c d\phi$; and this multiplied by (7) is the normal flux of force through the strip. Integrating the product from $\phi=0$ to $\phi=\pi$, we get the flux through the whole of the upper half of the ring. The result is simply

$$\frac{2\pi i c}{a}\left(L-\frac{5}{4}\right)\int_0^\pi (a-c\cos\phi)\sin\phi\, d\phi,$$

the second term in (7) being neglected because it gives a term of the third order. Thus this part of the flux is

$$4\pi ci\left(L-\frac{5}{4}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Adding this to (5), we have the whole flux sought, divided by i, equal to

$$\pi\left\{4a(L-2)+2c\left(L-\frac{5}{4}\right)-\frac{c^2}{16a}(2L+19)\right\}, \quad . \quad (9)$$

where $L=\log_e\frac{8a}{c}$. This, then, is the Coefficient of Self-Induction. If in this expression a and c are taken in centimetres, the result is the coefficient measured in absolute units; and if this is divided by 10^9, we have the coefficient of self-induction in secohms.

Thus, for example, a circular current running in a wire the diameter of whose cross-section is 2 millim., while the diameter of its central filament is 2 centim., has a coefficient of self-induction of about 59·207 absolute units, or $\frac{59{\cdot}207}{10^9}$ secohms; and if the dimensions of the cross-section were neglected, this number would be 58·866.

Clerk Maxwell (Elec. and Mag. vol. ii. art. 704) gives the coefficient of self-induction as $4\pi a(L-2)$, which agrees with (9) in the principal term.

In the same way we may find the coefficient for a superficial current in the wire. For (*antea*, p. 221) if q is the total quantity of the superficial current, we have the value of Gx at E equal to

$$2q\left[a(L-2)-\frac{c}{4}(2L-1)+\frac{c^2}{16a}\left(L+\frac{11}{4}\right)\right],$$

while from the value of the potential we find the normal flux of force through the upper half of the anchor-ring equal to

$$2\pi qc(2L+1).$$

Hence the Coefficient of Self-Induction is

$$\pi\left\{4a(L-2)+2c\left(L+\frac{3}{2}\right)+\frac{c^2}{16a}(4L+11)\right\}, \quad . \quad (10)$$

which is somewhat greater than the value (9) for a steady current.

XXXVII. *Note on a New Photometric Method, and a Photometer for same. By* Mr. J. B. Spurge.* (Abstract.)

The method consists in using two diffusing screens (illuminated respectively by the lights to be compared) as secondary sources, and adjusting to equality the luminosity of equidistant internal surfaces by varying the apertures through which the light passes from the screens to these surfaces. By reducing the sizes of the apertures, the author has been enabled to compare lights of different colour; for when of sufficiently feeble intensity, coloured lights are indistinguishable from white or grey.

The photometer is made up of two tubes mounted at 45° to an axis, about which one of them is capable of rotating. When in the same horizontal plane, the axes of the tubes form the sides of an isosceles right-angled triangle, at the middle of whose hypothenuse the light to be tested is placed; this illuminates one of the screens, whilst the standard light shines on the other. These screens, used as secondary sources, are situated a short distance away from the outer ends of the tubes, whilst the inner surfaces of the near ends of the tubes are viewed by means of a mirror. By turning the movable tube about the inclined axis, and rotating the source about a vertical axis, the illuminating effect of the source in any direction can be tested.

Discussion.

Capt. Abney said the law of inverse squares was not true for weak lights, for the proportions in which the light from sources of equal intensity had to be reduced to appear white or grey depended greatly on the colour, being much greater for violet than for red. Only for the yellow-green rays was the ordinary law of illumination true when the intensities were feeble.

Mr. Blakesley, Prof. S. P. Thompson, and the President also took part in the discussion.

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* Read January 26, 1894.

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PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1892-93.

February 12th, 1892.

Prof. W. E. Ayrton, Past President, in the Chair.

Mr. W. R. Bower and Mr. E. Edser were elected Members of the Society.

Professor S. P. Thompson, F.R.S., communicated a Note on Supplementary Colours.

February 26th, 1892.

Prof. W. E. Ayrton, Past President, in the Chair.

The following communications were made:—

"On Modes of representing Electromotive Forces and Currents in Diagrams." By Prof. S. P. Thompson, F.R.S.

"On the Flexure of Long Pillars under their own Weight." By Prof. M. FitzGerald.

March 5th, 1892.

Prof. S. P. Thompson, Vice-President, in the Chair.

Miss L. E. Walter was elected a Member of the Society.

The following communications were made:—

"Note on the Electromotive Forces of Gold and Platinum Cells." By Prof. E. F. Herroun.

"A new Instrument for showing the effect of Persistence of Vision." By E. Stuart Bruce, M.A.

"Some Electrical Instruments." By Mr. R. W. Paul.

March 11th, 1892.

Prof. A. W. Rücker, Vice-President, in the Chair.

The following communications were made:—

"A Thermodynamic View of the Action of Light upon Silver Chloride." By H. M. Elder, M.A.

"On Choking Coils." By Prof. J. Perry, F.R.S.

April 8th, 1892.

Dr. J. H. Gladstone, Past President, in the Chair.

Mr. C. T. Mitchell, Barrister-at-Law, was elected a Member of the Society.

The following communications were made:—

"On the Construction of a Colour-Map." By W. Baily, M.A.

"A Mnemonic Table for changing from Electrostatic to Practical and C.G.S. Electromagnetic Units." By W. Gleed, M.A.

"On the Law of Colour in relation to Chemical Constitution." By Mr. W. Ackroyd.

May 13th, 1892.

Dr. E. Atkinson, Treasurer, in the Chair.

Mr. Gerrans was elected a Member of the Society.

The following communications were made:—

"An Instrument for describing Parabolas." By Mr. R. Inwards.

"Some Electrical Instruments." By Mr. F. H. NALDER.

"A Portable Instrument for measuring Magnetic Fields." By Messrs. E. EDSER and H. STANSFIELD.

"On a Unit of Measurement of Light and Colour." By Mr. J. W. LOVIBOND.

Mr. R. W. PAUL exhibited a form of Wheatstone Bridge.

May 27th, 1892.

Mr. WALTER BAILY, Vice-President, in the Chair.

A communication, "On the Present State of our Knowledge of the Connexion between Ether and Matter," was made by Prof. O. J. LODGE, F.R.S.

June 10th, 1892.

Mr. WALTER BAILY, Vice-President, in the Chair.

The following communications were made:—

"On some Points connected with the Electromotive Force of Secondary Batteries." By Dr. GLADSTONE, F.R.S., and Mr. HIBBERT.

"On Workshop Ballistic and other Shielded Galvanometers." By Prof. W. E. AYRTON, F.R.S., and Mr. T. MATHER.

June 24th, 1892.

Prof. A. W. RÜCKER, Vice-President, in the Chair.

The following communications were made:—

"On Breath Figures." By Mr. W. B. CROFT.

"On the Measurement of the Internal Resistance of Cells." By Mr. E. WYTHE SMITH.

"On the Relation of the Dimensions of Physical Quantities to Directions in Space." By Mr. W. WILLIAMS.

"On the Laws of Molecular Force." By Mr. W. SUTHERLAND.

October 28th, 1892.

Dr. J. H. **Gladstone**, Past President, in the Chair.

Mr. G. H. **Wyatt** was elected a Member of the Society.

Discussions took place upon Mr. **Williams**'s paper "On the Relation of the Dimensions of Physical Quantities to Directions in Space," and upon Mr. **Sutherland**'s paper "On the Laws of Molecular Force;" in the course of which communications were made, "On the Determination of Critical Density," by Dr. **Young** and Mr. **Thomas**, "On the Determination of the Critical Volume" and "On the Boiling-points of Different Liquids at Equal Pressures," by Dr. **Young**.

November 11th, 1892.

Mr. **Walter Baily**, Vice-President, in the Chair.

Mr. **Porter** was elected a Member of the Society.

The Discussion on Mr. **Williams**'s paper "On the Relation of the Dimensions of Physical Quantities to Directions in Space" was continued.

November 25th, 1892.

Prof. S. P. **Thompson**, Vice-President, in the Chair.

Messrs. T. G. **Fry**, A. G. **Bessemer**, Junior, and E. F. **Fournier D'Albe** were elected Members of the Society.

A communication on "Experiments in Electric and Magnetic Fields, constant and varying," was made by Messrs. E. C. **Rimington** and **Wythe Smith**.

December 9th, 1892.

Mr. **Walter Baily**, Vice-President, in the Chair.

Major H. E. **Rawson**, R.E., was elected a Member of the Society.

The following communications were made:—

"On the Spectra of various Orders of Colours in Newton's Scale." By W. B. **Croft**, M.A.

"On the Diffusion of Light." By Dr. W. E. **Sumpner**.

January 13th, 1893.

Prof. G. F. **FitzGerald**, President, in the Chair.

Mr. **Haselfoot** was elected a Member of the Society.

A communication on "Science-Teaching" was made by Mr. F. W. **Sanderson**.

January 27th, 1893.

Mr. **Walter Baily**, Vice-President, in the Chair.

Mr. T. W. **Morton** was elected a Member of the Society.

The following communications were made:—

"On Japanese Magic Mirrors." By Prof. S. P. **Thompson**.

"On the Functions of the Retina." By Mr. W. F. **Stanley**.

Annual General Meeting.

February 10th, 1893.

Mr. WALTER BAILY, Vice-President, in the Chair.

The following Report of the Council was read by the Chairman :—

Since our last General Meeting there have been held fifteen ordinary business Meetings, in every case at the Laboratory of the Royal College of Science in South Kensington. As in the previous Report there was one, so now there are two more meetings to record than formerly was wont to be the case. This has been contrived by holding a meeting in October, though November has hitherto been the month for recommencing the meetings after the summer recess, and an extra meeting in January.

The Council has to mention the following deaths among the Members of the Society :—

The Rev. T. PELHAM DALE, M.A.; Mr. BENJAMIN LOEWY, F.R.A.S., one of the original Members of the Society; The DUKE OF MARLBOROUGH; Mr. C. E. WALDUCK; Mr. C. THOMPSON; Mr. P. W. WILLANS; Dr. HIRST; and on the 8th instant Mr. G. M. WHIPPLE, Member of our Council.

A new departure has been made which entails very much greater Secretarial work than heretofore. Most of our papers are now printed before being read, numerous proofs are sent to Members, and proofs and invitations are often sent to other scientific men who are likely to add, by their remarks, to the value of the discussions.

The numbers of the Society now amount to 371 Ordinary and 12 Honorary Members.

The following additions have been made to the Library during the year 1892–93 :—

Newspapers and Magazines :—

Nature.
The Electrical Review.
Engineering.
The Electrician.
Electrical Plant.
Ingeniero y Ferretero Español y Sud Americano.

Crónica Cientifica.
The Open Court.
Invention.
The Philosophical Magazine.
Beiblätter der Physik.
Annalen der Physik.
Journal de Physique.

Journals of Societies, British:—

Proceedings of the Royal Society.
Journal of the Society of Arts.
Journal of the Institute of Electrical Engineers.
Memoirs and Proceedings of the Manchester Literary and Philosophical Society.
Quarterly Journal of the Royal Meteorological Society.
Proceedings of the Institute of Mechanical Engineers.
Proceedings of the Birmingham Philosophical Society.

Journals of Societies, Colonial:—

Geological and Natural History Survey of Canada.
Proceedings and Transactions of the Nova Scotian Institute of Natural Science.
Hong Kong Observatory Observations.

Journals of Societies, Foreign:—

Proceedings of the Academy of Natural Sciences, Philadelphia.
Journal of the Physical and Chemical Society of Russia.
Mémoires and Séances of the Physical Society of France.
Journal of the College of Science of Japan.
Bulletin International de l'Académie des Sciences de Cracovie.

Books, &c.:—

North Atlantic Expedition. xxi.
Catalogue of Books added to Radcliffe Library, Oxford. 1891.
Sinopsis Mineralogica, by Carlos F. de Landero.

The Rev. T. **Pelham Dale**, M.A., was a son of the Rev. Thomas Dale, the well-known Canon of St. Paul's. He was born at Greenwich in 1821. He distinguished himself at Cambridge in the Mathematical Tripos, and became a Fellow of Sydney Sussex College. From 1847 to 1881 he was Rector of St. Vedast, Foster Lane, with St. Michael-le-Querne, in the City of London. In consequence of an action taken against him by his Churchwardens on a matter of ritual, he was committed to Holloway Prison for disobedience to the decision of the Court of Arches. On his release he retired to the quiet parish of Sansthorpe, in Lincolnshire, where he died last April. He was a man of varied attainments, and much interested in Mathematical and Physical Science. In early days he was occupied for a while in an engineer's workshop, and gave popular lectures and contributed to the 'Mechanics' Magazine.' In later life he was a Fellow of the Royal Astronomical Society, and served on its Council. About thirty-five years ago he took up the study of the Undulatory Theory of Light, and, in conjunction with Dr. Gladstone, he published several papers upon the subject of the Refraction of Light, two of which appeared in the 'Philosophical Transactions' for 1858 and 1863—one was on the "Influence of Temperature"; the other, "Researches on the Refraction, Dispersion, and Sensitiveness of Liquids." In later years he reverted to the subject in several papers communicated to this Society.

P. W. **Willans**, M.Inst.C.E., was killed suddenly on the 23rd May, 1892, at the age of 41, by being thrown from his carriage. Soon after his apprenticeship ended in 1874 he invented his famous high-speed steam-engine, and in the improvement and manufacture of his engines since that time he discovered for himself, and taught to mechanical engineers, many important matters in the application of thermodynamical principles; and there is a perfectly general acknowledgment among members of his profession that in doing this and in other ways his services to the steam-engine were as great as those of any man of this century. Electrical engineers in especial are under obligations to him.

Mr. Willans had excellences of ability and character rarely found united in one man. His perfect engineering training and clear-headed judgment enabled him to select and utilize whatever was valuable in the results of his long and costly thermodynamic researches. He will long be remembered for his courtesy and kind heart. He was never so happy as when assisting others less gifted than himself.

Dr. THOMAS ARCHER HIRST, F.R.S., F.R.A.S., who died on the 16th February, 1892, at 7 Oxford and Cambridge Mansions, W., was born in 1830 at Heckmondwike, Yorkshire, and early in life made the acquaintance of Professor Tyndall, and acquired those tastes for mathematics, physics, and chemistry, which enabled him to occupy a distinguished position in the scientific world. He carried on studies in these subjects at Marburg, Hesse-Cassel, Göttingen, Berlin, and Paris, and during his career he held many appointments, and gave to the world a number of important writings on scientific subjects. In 1861 he was elected a Fellow of the Royal Society, and he was also one of the original Members of the London Mathematical Society, which was founded in 1864, under the presidentship of Professor De Morgan, by students of University College. He was a Member of its Council continuously from 1864 to 1883, was Treasurer for several years, and its President from 1872 and 1874. In 1865 Dr. Hirst was appointed Professor of Mathematical Physics in University College, London, which chair he held until 1867, when he succeeded De Morgan as Professor of Mathematics in that College. In 1870 he accepted the newly created appointment of Assistant Registrar in the University of London, in consequence of which he resigned not only his professorship, but shortly afterwards his general secretaryship of the British Association, an office he had filled since the meeting at Nottingham in 1866. Early in 1873, when the Royal Naval College was founded at Greenwich, he became director of studies, and held the post for ten years. Since his health obliged him to retire from his position at Greenwich, he has passed several winters abroad. He has, however, published since then several papers, which have appeared in the 'Proceedings' of the Mathematical Society of London. Dr. Hirst was three times a Member of the Council of the Royal Society, and twice one of its Vice-Presidents. In 1883 one of the Royal Medals was awarded to him. He was, too, a Fellow of the Royal Astronomical Society, an *ex officio* Member of the Council of the British Association for the Advancement of Science, an Honorary Member of the Naturforschende Gesellschaft at Marburg, of that of Halle, of the Société Philomatique, Paris, and of the Philosophical Society, Cambridge. He served for some years on the Council of University College, London, and in 1882 was made a Fellow of the University of London.

The death took place on the 8th of February, 1893, at his residence in Richmond, of Mr. GEORGE MATHEWS WHIPPLE, Superintendent of the Kew Observatory. Mr. Whipple had suffered from a painful internal complaint, but until a fortnight ago he was approaching convalescence, and his death was therefore unexpected. He was 50 years of age. He entered the Kew Observatory in 1858, and in 1862 became magnetic assistant. In 1866 the first set of results of sun-spot measurements was published, under the title of 'Researches on Solar Physics.' Most of the measurements were Mr. Whipple's, and subsequent editions of the work were under his direction. Mr. Whipple made a special study of wind-force and wind-velocities. He was, throughout the greater part of his life, constantly carrying on experiments with a view to determine wind-force, and to find out what were the best instruments for securing accuracy of results. In 1874 he made a series of experiments at the Crystal Palace, with the object of finding the constants of Robinson's anemometers; these were afterwards continued in another form, in conjunction with Mr. Dines. Mr. Whipple designed the apparatus used for testing the dark shades of sextants and other optical instruments; and also introduced several improvements in the Kew pattern magnetic instruments. In 1873 he was engaged with Captain Heaviside in making a series of pendulum-experiments to determine the force of gravity. In 1881 these experiments were repeated in association with Major Herschel, F.R.S., and in 1888 with General Walker, F.R.S. He contributed a number of important papers to the Royal Society. He was appointed Superintendent of Kew Observatory in 1876, and for the past 16 years he has also contributed results of investigations to the Royal Meteorological Society and to photographic publications. The magnetic part of the report of the Committee of the Royal Society which met to discuss the Krakatoa eruption and the subsequent phenomena were written by him.

The adoption of the Report was proposed and passed unanimously.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows:—

President.—Prof. A. W. RÜCKER, M.A., F.R.S.

Vice-Presidents who have filled the Office of President.—Dr. J. H. GLADSTONE, F.R.S.; Prof. G. C. FOSTER, F.R.S.; Prof. W. G. ADAMS, M.A., F.R.S.; The Lord KELVIN, D.C.L., LL.D., P.R.S.; Prof. R. B. CLIFTON, M.A., F.R.S.; Prof. A. W. REINOLD, M.A., F.R.S.; Prof. W. E. AYRTON, F.R.S.; Prof. G. F. FITZGERALD, M.A., F.R.S.

Vice-Presidents.—WALTER BAILY, M.A.; Major-General E. R. FESTING, R.E., F.R.S.; Prof. J. PERRY, D.Sc., F.R.S.; Prof. S. P. THOMPSON, D.Sc., F.R.S.

Secretaries.—H. M. ELDER, M.A.; T. H. BLAKESLEY, M.A., M.Inst.C.E.

Treasurer.—Dr. E. ATKINSON.

Demonstrator and Librarian.—C. VERNON BOYS, F.R.S.

Other Members of Council.—SHELFORD BIDWELL, M.A., LL.B., F.R.S.; L. FLETCHER, M.A., F.R.S.; Prof. G. FULLER; Prof. O. HENRICI, Ph.D., F.R.S.; Prof. J. V. JONES, M.A.; Prof. G. M. MINCHIN, M.A.; Rev. F. J. SMITH, M.A.; W. E. SUMPNER, D.Sc.; J. SWINBURNE; JAMES WIMSHURST.

Votes of thanks were passed to the Lords Committee of the Council on Education; to the OFFICERS; and to the AUDITORS.

The Treasurer in Account with the Physical Society, from January 1st, 1892, to December 31st, 1892.

Dr.	£ s. d.	£ s. d.
Balance in Bank, January 1, 1892		552 10 4
Entrance-Fees	19 0 0	
Life Compositions	80 0 0	
Subscriptions for 1889 and before	13 0 0	
" 1890	11 0 0	
" 1891	28 0 0	
" 1892	136 0 0	
" 1893	6 0 0	
		293 0 0
One year's Dividend on £400 Furness 4 per cent. Debenture Stock, less Income Tax	15 12 0	
One year's Dividend on £200 Lancaster Corporation Stock, less Income Tax	7 16 0	
One year's Dividend on £575 Midland 4 per cent. Preference Stock, less Income Tax	22 8 6	
One year's Dividend on £200 Metropolitan Board of Works Stock, less Income Tax	6 16 8	
One year's Dividend on £254$\frac{3}{5}$ New South Wales 3½ per cent. Stock, less Income Tax	8 13 6	
		61 6 8
Sale of Proceedings, Books, and Memoirs		43 12 6
Balance due to Treasurer		8 7 8
		£958 17 2

Cr.	£ s. d.	£ s. d.
Balance due to Treasurer		1 0 0
Excess Subscriptions returned		0 6 0
Taylor and Francis:—		
Proceedings, vol. ii. (parts 2, 3, and 4)	130 11 6	
Physical Memoirs	2 6 6	
Postage &c. of Proceedings	13 17 8	
Members' separate copies	24 1 6	
Miscellaneous printing	46 16 6	
		217 13 8
Gyde (Binding)		4 4 3
Haggard (Binding)		0 12 9
Williams and Norgate (Periodicals)		3 11 0
Stationery		1 3 6
Reporting		23 4 4
Attendance at Meetings and petty expenses		7 9 1
Electric Light		0 11 6
Purchase of £325 Midland 4 per cent. Perpetual Preference Stock	431 18 3	
Less Dividend	6 6 9	
		425 11 6
Bank Charges		0 3 7
Balance in Bank		266 13 10
Petty Cash:—		
Mr. Elder	4 10 1	
Mr. Blakesley	0 16 10	
Dr. Atkinson	1 5 3	
		6 12 2
		£958 17 2

Audited and found correct,

HARRY M. ELDER,
ALEXANDER PELHAM TROTTER, } *Auditors.*

February 4, 1893

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, December 31, 1892.

Assets.	£	s.	d.	Liabilities.	£	s.	d.
Balance in Bank	266	13	10	Subscriptions in advance	6	0	0
Subscriptions due, estimated to produce	50	0	0				
£400 Furness 4 per cent. Debenture Stock	456	0	0				
£900 Midland 4 per cent. Preference Stock	1075	11	6				
£200 Metropolitan Board of Works Stock	217	0	0				
£200 Lancaster Corporation 4 per cent. Stock	224	0	0				
£254⅜ New South Wales 3½ per cent. Stock	249	7	5	Balance	2532	12	9
	£2538	12	9		£2538	12	9

Audited and found correct,

HARRY M. ELDER,
ALEXANDER PELHAM TROTTER, } *Auditors.*

February 4, 1893.

PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1893-94.

February 10th, 1893.

WALTER BAILY, M.A., Vice-President, in the Chair.

Mr. C. E. GROVE was elected a Member of the Society.

Dr. J. H. GLADSTONE, F.R.S., read a paper on "Some Recent Determinations of Molecular Refraction and Dispersion."

Mr. E. C. C. BALY read a paper on "Separation and Striation of Rarefied Gases under the Influence of the Electric Discharge."

February 24th, 1893.

Prof. A. W. RÜCKER, President, in the Chair.

Prof. J. D. EVERETT, F.R.S., communicated a paper on "A new and handy Focometer."

A paper by G. H. BRYAN, M.A., on "A Hydrodynamical Proof of the Equations of Motion of a Perforated Solid, with Applications to the Motion of a Fine Rigid Framework in Circulating Liquid," was read by Dr. C. V. BURTON.

Appended to the paper is a note by Dr. BURTON relating to the proper measurement of the impulse of cyclic motion; and in a further note Mr. **Bryan** shows how the equations of motion may be deduced from Dr. Burton's suggestions.

Dr. C. V. **Burton** read a paper on "Plane and Spherical Sound-Waves of Finite Amplitude."

March 10th, 1893.

Prof. A. W. **Rücker**, President, in the Chair.

Mr. W. **Williams** was elected a Member of the Society.

Dr. C. V. **Burton** read a paper "On the Applicability of Lagrange's Equations of Motion in a general class of Problems, with especial reference to the Motion of a Perforated Solid in a Liquid."

Prof. G. M. **Minchin** read a paper "On the Magnetic Field of a Circular Current."

March 24th, 1893.

Prof. A. W. **Rücker**, President, in the Chair.

Some excellent photographs of flying bullets and of the air-waves produced by vibrating hammers were exhibited, the originals of which had been taken by Prof. **Mach**.

The following communications were made:—

"On the Differential Equation of Electrical Flow." By **Thomas H. Blakesley**, M.A.

"On the Viscosity of Liquids." By Prof. J. **Perry**, F.R.S., assisted by J. **Graham**, B.A., and Mr. C. W. **Heath**.

April 28th, 1893.

Prof. W. E. **Ayrton**, Past President, in the Chair.

Mr. F. **Harrison** was elected a Member of the Society.

A discussion took place upon the subject of the paper by Prof. **Perry**, read March 24th, on the Viscosity of Liquids.

Prof. **Perry** read a communication he had received from Prof. **Maurice Fitzgerald** on the subject, in which the latter discussed the corrections necessary for reducing the results obtained by circular motion to the corresponding motion in plane layers. He shows that, in addition to the circular motion, the effect is complicated by radial flow, due to "centrifugal head," which causes the liquid to pass outwards near the bottom of the trough, and inwards across the edge of the suspended cylinder, with continuations along the sides of the trough and cylinder. Taking this motion into account, the formula

$$v=\mathrm{A}r^{1+\frac{c}{\mu}}+\frac{\mathrm{B}}{r}$$

is deduced, where v is the velocity, μ the viscosity, A and B arbitrary constants, and c a constant depending on the radial flow. When $c=0$ the formula reduces to equation (5) of the paper, whilst if $c=-2\mu$ it becomes

$$v=\frac{\mathrm{C}}{r}.$$

The subject of critical velocities in non-turbulent motion is referred to, and some probable effects of the anomalous variations of density and viscosity of sperm-oil, noticed by the authors of the paper, are pointed out.

Prof. Perry, in further reply to Prof. Osborne Reynolds's comments, said he understood Prof. Reynolds to have proved that friction was proportional to velocity when the motion was steady. Experiments he (Prof. Perry) had made with disks of iron and glass in revolving mercury seemed to show that this was not the case. On replacing the mercury by sperm-oil he found that up to a certain speed friction was strictly proportional to velocity, whilst above that speed friction varied as $v^{1\cdot 25}$. Coloured streaks in the liquid remained unbroken even at the highest speeds. He therefore concluded that continuity of the streaks was not necessarily accompanied by a linear law of friction.

Mr. E. C. Rimington read a paper on "Luminous Discharges in Electrodeless Vacuum-Tubes."

May 12th, 1893.

Prof. A. W. Rücker, President, in the Chair.

The following communications were made:—

"On the Drawing of Curves by their Curvature." By C. V. Boys, A.R.S.M., F.R.S.

"On the Foundations of Dynamics." By Prof. O. J. Lodge, F.R.S.

May 26th, 1893.

Prof. A. W. **Rücker**, President, in the Chair.

Mr. C. J. **Woodward** showed some "Experiments with a Vibrating Bar."

On suspending the bar by two loops of cord, and placing it over a resonance box, the sound was greatly intensified. When placed crosswise, and partly over the box, a position could be found where no increase of sound resulted, whilst a little movement in either direction from this position caused a considerable increase.

The discussion on Dr. **Lodge's** paper, "The Foundations of Dynamics," was then resumed.

Communications on the subject from Mr. S. H. **Burbury**, Dr. G. **Johnstone Stoney**, and Prof. E. F. **Herroun** were read.

June 9th, 1893.

Prof. J. **Perry**, F.R.S., Vice-President, in the Chair.

Prof. A. **Jamieson**, Prof. H. **Stroud**, and Mr. J. **Wade** were elected Members of the Society.

The following communications were made:—

"A new Photometer." By Mr. A. P. **Trotter**.

"Some Notes on Photometry." By Prof. S. P. **Thompson**, F.R.S.

"The Magnetic Field close to the Surface of a Wire conveying an Electric Current." By Prof. G. M. **Minchin**, M.A.

June 23rd, 1893.

Prof. A. W. **Rücker**, President, in the Chair.

Prof. J. **Cox**, M.A., was elected a Member of the Society.

Mr. F. H. **Nalder** exhibited a Bridge and Commutator for comparing Resistances by Prof. Carey Foster's Method.

Mr. R. W. **Paul** exhibited a compact form of Sulphuric-acid Voltameter of small resistance. The voltameter is a modification of a pattern designed at the Central Institution, in which the rate of decomposition is determined from the time required to fill a bulb made in the stem of a thistle funnel. He also showed a handy form of Daniell cell, devised by Prof. Barrett. When not in use, the porous pot containing the zinc is removed from the copper sulphate solution, and placed in a vessel containing zinc sulphate or sulphuric acid.

Mr. W. R. **Pidgeon** and Mr. **James Wimshurst** each read a paper on "An Influence Machine," and exhibited their machines in action.

A paper on "A new Volumenometer," by Mr. J. E. **Myers**, describing the developed form of Prof. Stroud's instrument, was, in the absence of the Author, taken as read.

A paper on "Long-distance Telephony," by Prof. J. **Perry**, F.R.S., assisted by H. A. **Beeston**, was read by the former.

October 27th, 1893.

Prof. J. **Perry**, F.R.S., Vice-President, in the Chair.

Mr. E. C. **Rimington** read a paper "On the Behaviour of an Air-Core Transformer when the Frequency is below a certain Critical Value."

November 10th, 1893.

Prof. A. W. **Rücker**, President, in the Chair.

Mr. R. S. **Cole** was elected a Member of the Society.

The following communications were made:—

"The Separation of Three Liquids by Fractional Distillation." By Prof. F. R. **Barrell**, Mr. G. L. **Thomas**, and Prof. S. **Young**.

"Note on the Generalizations of Van der Waals regarding Corresponding Temperatures, Pressures, and Volumes." By Prof. S. **Young**.

An Instrument for drawing Conic Sections was exhibited and described by Mr. J. **Gillett**, B.A.

This consists of a spindle inclined to a plane board, and a tube fixed to the spindle at an angle. A pencil which passes through the tube traces out a cone in space as the spindle is turned, and on sliding the pencil through the tube so as to keep its point against the plane, the point traces out a conic—the section of the cone made by the plane of the board. A circle, ellipse, parabole, or hyperbole can be drawn according to the inclination of the spindle to the board.

Prof. **Henrici** said a similar instrument had been described in an Arabian manuscript 1000 years old, and had been independently re-invented by both a German and an Italian mathematician. He thought the fact of the angle between the spindle and the tube in Mr. Gillett's instrument not being adjustable was a disadvantage.

Mr. **Inwards** and Prof. **Herschel** also took part in the discussion, to which Mr. **Gillett** replied.

November 24th, 1893.

Prof. A. W. **Rücker**, President, in the Chair.

Col. **Maitland**, C.B., was elected a Member of the Society.

Prof. S. P. **Thompson** occupied the Chair while the President read a paper "On the Magnetic Shielding of Concentric Spherical Shells."

Prof. G. M. **Minchin**, M.A., read a paper "On the Action of Electromagnetic Radiations on Films containing Metallic Powders."

December 8th, 1893.

Prof. A. W. **Rücker**, President, in the Chair.

Mr. J. A. **Gillett** and Mr. F. **Hovenden** were elected Members of the Society.

A paper by Mr. **James Swinburne**, "On a Potentiometer for Alternating Currents," was read by the Hon. Secretary.

Prof. G. M. **Minchin**, M.A., made communications "On the Calculation of the Coefficient of Self-Induction of a Circular Current of given Aperture and Cross-Section," and "On the Magnetic Field of a Current running in a Cylindrical Coil."

January 26th, 1894.

Prof. A. W. **Rücker**, President, in the Chair.

The following communications were made:—

"On a new Mode of making Magic Mirrors." By Mr. J. W. **Kearton**.

"Some Observations on Diffraction." By Mr. W. B. **Croft**.

"A new Photometric Method, and Photometer for same." By Mr. J. W. **Spurge**.

Annual General Meeting.

February 9th, 1894.

Prof. A. W. Rücker, President, in the Chair.

The following Report of the Council was read by the Chairman:—

The Council has this year to congratulate the Society upon the completion of twenty years of useful existence. The foundation took place in 1874; and it is a satisfaction to know that a very large proportion of the founders is still alive and as enthusiastic for the welfare of the Society as a score of years ago.

During this time the Society has published nearly twelve volumes of 'Proceedings,' besides translations from foreign sources of the important Physical Memoirs of Helmholtz, Hittorf, Puluj, and Van der Waals, as well as the entire Scientific Works of Joule and of Wheatstone.

The Council has never made the mistake of refusing any communication which has turned out eventually to have embodied matters of genuine interest in Physics, and every new idea of importance in this field has always found a welcome reception and an early public utterance in England at the meetings of this Society.

Of late the Council of the Physical Society has been the first body to give expression to the desire generally felt by the followers of Physical Science that some single central and widely inclusive publication of Physical work should be set on foot and issued periodically in the English language. It is felt that such a work would be of the utmost value to scientific men; and though the idea is perhaps still in the nebulous condition, it may be of interest to the Members generally to know that the Council has taken the initiative in making its opinions upon this subject known in quarters where it is hoped some cooperation may be afforded.

The Society has to regret the loss by death of Professor John Tyndall and of Mr. Paice.

The world as well as this country, where he was lately a visitor, must have heard with the keenest sorrow of the death of Professor Hertz.

Since the last Annual General Meeting there have been fourteen Ordinary Meetings.

The following additions have been made to the Library during 1893–94:—

Newspapers and Magazines:—

Nature.
Electrical Review.
Electrician.
Electrical Plant.
Ingeniero y Ferretero Español y Sud Americano.
Crónica Cientifica.
Invention.
The Open Court.
The Iron, Steel, and Coal Times.
Engineering.
Industries and Iron.
The Philosophical Magazine.
Beiblätter der Physik.
Annalen der Physik.
Journal de Physique.

Journals of Societies, British:—

Journal of the Society of Arts.
Proceedings of the Birmingham Philosophical Society.
Transactions and Proceedings of the Cambridge Philosophical Society.
Transactions and Proceedings of the Royal Dublin Society.
Journal of the Institute of Electrical Engineers.
Memoirs and Proceedings of the Manchester Literary and Philosophical Society.
Proceedings of the Institute of Mechanical Engineers.
Quarterly Journal of the Royal Meteorological Society.
Proceedings of the Royal Institution.
Proceedings of the Royal Society.
Proceedings of the Edinburgh Mathematical Society.
Journal of the Camera Club.

Journals of Societies, Colonial:—

Journal and Proceedings of the Royal Society of New South Wales.
Geological and Natural History Survey of Canada: Annual Report and Maps.

Hong Kong Observatory Observations.

Proceedings and Transactions of the Nova Scotia Institute of Natural Science.

Transactions of the Canadian Institute, and 5th Annual Report.

Papers of the Mathematical and Physical Society of Toronto.

Journals of Societies, Foreign:—

Journal of the Physical and Chemical Society of Russia.

Verhandlungen, Berlin Physical Society.

Mémoires and Séances of the Physical Society of France.

Journal of the College of Science, Japan.

Transactions of the Seismological Society, Japan.

Seances and Bulletin International de l'Académie des Sciences, Cracovie.

Prace Matematyczno-Fizyczna, Warsaw.

Report of the Ohio Meteorological Bureau.

Technology Quarterly and Proceedings of the Society of Arts, Massachusetts.

Report of the Ohio Weather and Crop Service.

Proceedings of the Academy of Natural Sciences, Philadelphia.

Books, &c.:—

Waterdale Researches, by Waterdale.

Inertia of Matter, by Ernest Major.

Œuvres de Laplace, Tome ix.

Smithsonian Report, 1890.

Greenwich Magnetic and Meteorological Observations, 1890.

Greenwich Spectroscopic and Photographic Results, 1890.

Personal Recollections of Werner von Siemens.

University College Calendar.

University of Japan Calendar.

Catalogue of the Stratigraphical Collection of Canadian Rocks of the Chicago Exhibition.

Electricity in the Service of Man; translated from the German by Dr. Wormell.

In the death of Professor TYNDALL, which took place on Dec. 7, 1893, the Society has suffered the loss of one of its most distinguished Members, and English Science one of its most prominent representatives. Though he very rarely attended our Meetings, he warmly approved of the objects for which our Society was founded; he joined almost at the outset, and became a Life Member.

John Tyndall was born near Carlow, in Ireland, of parents in a humble rank of life, his father having been in the old Inland Revenue Police, with which he combined a small business as leather-dealer. There are grounds for believing that he is of the same stock as William Tindale, the first English translator of the Bible, and that his ancestors, with other cloth-workers, migrated from Gloucestershire to Ireland in the 17th century.

Tyndall's father was a strong Protestant, and Tyndall's early days, spent in the midst of Roman Catholic surroundings, accustomed him to religious controversy, and had much to do with the ability he afterwards developed in defending any position which he took up.

Up to the age of nineteen years he attended a local school, the master of which was Mr. Cornill, a Roman Catholic, for whom Tyndall always entertained respect and regard. On leaving school he obtained an appointment as Civil Assistant in the Ordnance Survey at a very small salary, which, however, with his abstemious habits, was sufficient for his modest needs.

The experience and knowledge Tyndall had gained in the Ordnance Survey was of great service, and on leaving he had no difficulty in getting employment as a Civil Engineer; he was engaged by a Manchester firm of Engineers, and was occupied in surveying and laying-out lines of railway in Lancashire and Yorkshire in that period of busy activity which culminated in the Railway Mania of 1846. Much of his time was spent in the neighbourhood of Preston, and, working hard as he did at his professional work, he yet found time for private study and other forms of mental improvement.

There he formed the acquaintance of Mr. George Edmondson, a member of the Society of Friends, who had a large and successful school at Tulketh Hall, near Preston, conducted on modern principles. On the failure of the Community formed by Robert Owen in Hampshire, the buildings were found to be admirably adapted for a school. Mr. Edmondson was induced to transfer his school thither. This was done in 1846, and the establishment was opened, under the name of Queenwood College, as a School and an Agriculture and Technical College, for which a large quantity of land formerly belonging to the Community was available.

Tyndall was in 1847 appointed as Teacher of Mathematics and Surveying, and Dr. Frankland of Chemistry and Physics, in the new undertaking. The difference in the ages of the Agricultural pupils and of the School boys made it difficult to work the two together, and it was ultimately decided to abandon the Agricultural Department. The School then began to improve. Mr. Edmondson, like many of the religious body to which he belonged, had great zeal and enthusiasm for education, quite fully alive to modern ideas, and with great capacity for organization. The School realized more than forty years ago what many are now wishing for, namely, a place in which Mathematics, the Sciences, and Modern Languages occupied the dominant position, Classics a subordinate one. The aims of the School were advanced; not less remarkable were some of the teaching staff engaged to carry them out in the department of Mathematics and the Sciences: it will be sufficient to mention only the names of Tyndall and of Hirst, whose loss this Society had last year to deplore, of Frankland and Debus. Which among our richly endowed public schools can show such a record as this? As was remarked by one who well knew Queenwood, "Only when we have seen other places do we fully realize how good a school Queenwood was."

During his stay at Queenwood Tyndall worked at Chemistry under his colleague Frankland, who in turn had the advantage of Tyndall's teaching in Mathematics.

After about a year at Queenwood, Tyndall went with his colleague Frankland to the University of Marburg, attracted by the fame of our distinguished and venerable honorary Member, Bunsen. There he worked at Chemistry, Mathematics, and Physics, but gave himself more to the latter subjects; and on taking his degree as Doctor of Philosophy, the subject of the Dissertation, which it was necessary to present for that purpose, was purely mathematical—"On a screw surface with inclined generatrix."

After a short visit to England in 1850 he returned to Marburg, accompanied by his friend Hirst. Previous to his return he called upon Dr. Francis, then, as now, Editor of the 'Philosophical Magazine,' and the result of this was that he entered into an engagement to furnish for that journal a series of reports on the Progress of the Physical Sciences in Germany; the first of these appeared in March 1851, and gave an account of recent researches on Electromagnetism.

Tyndall's first original work was one carried out in conjunction with Prof. Knoblauch, of Marburg, Extraordinary Professor of Physics there; it appeared in the 'Philosophical Magazine' for March 1850, under the title "On the Deportment of Crystalline Bodies between the Poles of a Magnet."

In this year he carried on by himself a research on the laws of magnetism in the laboratory of his friend Knoblauch, and published a paper on the subject in the 'Philosophical Magazine.' He went later to Berlin, where he worked in 1851 in the laboratory of Magnus, and here he made the acquaintance of the leading German Physicists, such as Du Bois-Reymond, Poggendorff, and Clausius.

In 1851 he returned to England, resuming his appointment at Queenwood, and in 1851 published a paper on the Polarity of Bismuth, the experiments for which were made in the laboratory of Magnus. He continued his researches at Queenwood, contributing at the same time his reports and also translations to the 'Philosophical Magazine.' These reports were eagerly welcomed by the English Physicists of that day, notably by Faraday and Grove. The interest which they aroused, and the high opinion held of him by the Physicists of Berlin (which Dr. Bence Jones, the Secretary of the Royal Institution, learnt of on a visit to Berlin), led to his being invited to give a Friday evening Lecture at the Royal Institution. The subject of this Lecture was on the "Influence of Material Aggregation on the Manipulation of Force." As our first President, Dr. Gladstone, observes—"The lecture was characterized by great *élan*, and a fine display of experimental powers, and seemed to take the audience by storm." Dr. Frankland says—"This lecture, though upon such an abstruse subject, took his audience, most popular as it was, by storm, and he received at the close quite an ovation."

The success of that lecture was such that, on the proposition of Faraday, he was shortly afterwards appointed Professor of Natural Philosophy in the Royal Institution. He left Queenwood to enter on his new duties there in July 1853, and was succeeded in his appointment by Dr. Hirst.

From that time his career has been before the world.

Tyndall's scientific work ranged over a wide extent of physical science. His thesis for his doctorate at Marburg was an essay on Screw-surfaces, a more mathematical topic than any which he subsequently attempted. His first physical paper was on the phenomena of a water-jet, in which he showed that the noises of streams and waterfalls are due to the entanglement of air. Before he left Marburg, the work which he had undertaken in conjunction with Knoblauch, on the magneto-optic properties of crystals and the relation of magnetism to molecular arrangement, was ready for publication. His second research on Diamagnetism was made in Berlin, where he worked under Magnus. In 1853, when he returned to London, he lectured on these researches at the Royal Institution, and in May of that year was chosen Professor of Natural Philosophy. For the next year or two he continued to work at Magnetism, and investigated the effect of pressure and of cleavage upon the magnetic and diamagnetic properties of matter. In the course of this work he was led to study glacier problems. After the publication, in 1855, of his essay on Slaty Cleavage he visited Switzerland in company with Professor Huxley, to investigate glacial action: and in 1860 he published his work on the Glaciers of the Alps. His mountaineering exploits in subsequent years are known to all the world. While at Marburg he had also worked at the phenomena of Heat, which formed the subject of prolonged researches. In 1852 he had sent to the Royal Society a paper on the conduction of heat through organic substances, such as wood, in which he proved the axis of maximum conductivity to coincide with the axis of maximum cohesion—namely, along the grain. But in 1859 he passed on to the study of the radiation of heat, and of the action of gases and vapours on the same. For some ten years he continued these researches. The Bakerian lecture for 1861 was his first formal memoir on this investigation. Using the thermopile as the measurer of radiation, and tubes closed by plates of rock-salt, he sought to unravel the absorptive powers which different vaporous media exert upon radiations of different sorts. He found the elementary and simple gases to absorb far less than those of more complex structure, the absorption increasing with the complexity of the molecular structure. The radiating powers present a corresponding contrast. Essences were found to produce unexpectedly great absorption. As with his researches on Diamagnetism, so with those on Radiant Heat, he collected the substance of his scattered memoirs into a volume which was published in convenient form for reference. In 1863 his more popular work 'Heat a mode of Motion' dealt mainly with elementary thermodynamics, and did useful service in bringing that subject to the level of comprehension of the ordinary educated reader. A like service was performed by his other popular works on Sound and on Light, the latter being a reproduction of the lectures which he delivered in America in 1872. These lectures were a brilliant experimental exposition of physical optics on a scale never before attempted. Tyndall also published two small volumes of notes on Electricity; but they were little more than synopses of his lectures, and did not aim at originality. From about the year 1875 Tyndall's attention was taken up with the germ theory of putrefaction, which he investigated with characteristic zest, employing as an observing agent a powerful beam of light, to detect the floating germs. It was in the course of these experiments that Tyndall discovered that to produce complete sterilization of the infusions employed it would not suffice to heat to boiling-point once, but that repeated raising to the boiling-point at intervals was infallibly successful. Till about 1883 he was actively occupied with this work; and he resumed the subject at intervals till the final breakdown of his health in 1886. From 1850 to 1883 he published no fewer than 135 memoirs and papers, beside numerous scattered Magazine articles and addresses, many of which were republished in the two volumes entitled 'Fragments of Science.' No account of Tyndall's activities would be complete which did not contain some reference to the numerous controversies into which he entered with various scientific men. He was an uncompromising adversary even in purely scientific topics. His dispute with Prof. J. D. Forbes on the origin of glacial motion, and that with Prof. Tait as to the merits of Robert Julius Mayer as the founder, rather than Joule, of the dynamical theory of heat, witness to this personal trait. His controversy with Prof. W. Thomson (now Lord Kelvin) on the subject of diamagnetic polarity was a curious episode, since the opinion, which seems to have been held at the time, that Tyndall had answered his critic's strictures would not now be upheld by any competent authority. His anti-theological utterances of the seventies, and his political diatribes of the past seven years, though intensely characteristic of the man, are no part of his scientific work. He retired after the breakdown in health of 1886 to the solitude of his residence at Hindhead, where he died on December 4, 1893.

Great as were his abilities as an investigator they were surpassed by his powers as an expounder. His public lectures were models of lucid exposition; and his patience and skill in the performance of difficult experiments unrivalled.

The adoption of the Report was proposed and passed unanimously.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows:—

President.—Prof. A. W. Rücker, M.A., F.R.S.

Vice-Presidents who have filled the Office of President.—Dr. J. H. Gladstone, F.R.S.; Prof. G. C. Foster, F.R.S.; Prof. W. G. Adams, M.A., F.R.S.; The Lord Kelvin, D.C.L., LL.D., P.R.S.; Prof. R. B. Clifton, M.A., F.R.S.; Prof. A. W. Reinold, M.A., F.R.S.; Prof. W. E. Ayrton, F.R.S.; Prof. G. F. Fitzgerald, M.A., F.R.S.

Vice-Presidents.—Walter Baily, M.A.; Major-General E. R. Festing, R.E., F.R.S.; Prof. J. Perry, D.Sc., F.R.S.; Prof. S. P. Thompson, D.Sc., F.R.S.

Secretaries.—H. M. Elder, M.A.; T. H. Blakesley, M.A., M.Inst.C.E.

Treasurer.—Dr. E. Atkinson.

Demonstrator and Librarian.—C. Vernon Boys, F.R.S.

Other Members of Council.—Shelford Bidwell, M.A., LL.B., F.R.S.; W. E. Sumpner, D.Sc.; Prof. G. Fuller; J. Swinburne; G. Johnstone Stoney, M.A., F.R.S.; R. E. Baynes, M.A.; Prof. G. M. Minchin, M.A.; L. Fletcher, M.A., F.R.S.; Prof. O. Henrici, Ph.D., F.R.S.; Prof. S. Young, D.Sc., F.R.S.

Votes of thanks were passed to the Lords Committee of the Council on Education; to the Officers; and to the Auditors.

THE TREASURER IN ACCOUNT WITH THE PHYSICAL SOCIETY, FROM JANUARY 1ST, 1893, TO DECEMBER 31ST, 1893.

Dr.	£	s.	d.	£	s.	d.
Balance in Bank, January 1, 1893				266	13	10
Entrance-Fees	10	0	0			
Life Compositions	60	0	0			
Subscriptions for 1891	6	0	0			
" 1892	21	0	0			
" 1893	123	0	0			
" 1894	4	0	0			
				224	0	0
Dividends, less Income Tax:—						
Furness 4 per cent. Debenture Stock	15	11	8			
Lancaster Corporation Stock *	3	17	10			
Midland 4 per cent. Preference Stock	35	1	3			
New South Wales 3½ per cent. Stock	8	13	1			
Metropolitan Board of Works Stock	6	16	4			
				70	0	2
Sale of Proceedings, Books, and Memoirs				32	7	9
Balance due to Treasurer				2	18	0
				£595	19	9

Cr.	£	s.	d.	£	s.	d.
Balance due to Treasurer				8	7	8
Taylor and Francis:—						
Proceedings, vol. xii. (parts 1 and 2)	81	12	0			
Members' separate copies	15	10	6			
Miscellaneous printing	47	9	9			
Postage	9	13	9			
				154	6	0
Gyde (Binding)				2	0	0
Haggard (Binding)	6	15	0			
Hoxey (Binding)	5	2	0			
Williams and Norgate	2	18	9			
				14	15	9
Attendance and petty expenses				8	3	9
Purchase of £100 Midland Preference Stock				134	6	9
Bank Charges				0	1	0
Petty Cash:—						
Mr. Elder	2	5	0			
Dr. Atkinson	1	13	0			
				3	18	0
Reporting				30	10	11
Balance in Bank				239	9	11
				£595	19	9

* Second half-year's dividend delayed at Bankers; will appear in 1894 accounts.

EDMUND ATKINSON, *Treasurer.*

Audited and found correct,

A. P. TROTTER,
RICHARD INWARDS, } *Auditors.*

February 5th, 1894.

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, December 31, 1893.

Assets.	£	s.	d.	Liabilities.	£	s.	d.
Balance at Bank	239	9	11	Subscriptions received in advance	4	0	0
Subscriptions due, estimated	28	0	0				
£400 Furness 4 per cent. Debenture Stock	456	0	0				
£1000 Midland 5 per cent. Preference Stock	1194	0	0				
£200 Metropolitan Board of Works Stock	217	0	0				
£200 Lancaster Corporation 4 per cent. Stock	224	0	0				
£254¾ New South Wales 3½ per cent. Stock	249	7	5	Balance	2603	17	4
	£2607	17	4		£2607	17	4

EDMUND ATKINSON, *Treasurer.*

Audited and found correct,

A. P. TROTTER,
RICHARD INWARDS. } *Auditors.*

February 5th, 1894.